Question 1 (100 total points)

A. (40 points) Consider a version of the standard consumption-savings problem, where a consumer seeks to solve:

$$\max_{\{c_t, a_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$c_t + a_{t+1} = Ra_t + y_t$$

where a_t are assets at date t, y_t is (exogenous) labor income y_t, and a_0, y_0 are given. The agent faces the constraints:

$$c_t \geq 0, \quad a_t \geq \underline{\mathrm{a}}.$$

Labor income y is stochastic, taking on values in compact set Y, and follows a Markov process with transition function Q.

(a) Write the Bellman equation for the consumer's problem characterizing the value function v. What conditions guarantee that this Bellman equation has a unique solution? What conditions guarantee that v is concave? What conditions guarantee that v is differentiable?

(b) Find the optimality conditions from the consumer's problem, paying special attention to the constraints.

(c) Suppose that $\beta R = 1$ and $u(c) = -\frac{1}{2}(c - \bar{c})^2$. If the borrowing constraint never binds, say $\underline{\mathrm{a}} = -\infty$, characterize the optimal consumption policy and the consumption dynamics.

(d) Now suppose that the borrowing constraint occasionally binds, $\underline{\mathrm{a}} < -\infty$, and characterize the consumption dynamics.

B. (30 points) Consider a simple search model of the housing market. Time is continuous, and there is a population of agents of mass 1 who are risk neutral and discount at rate r. A fraction H of the population owns (indivisible) houses, which are identical and yield flow utility u to their owners each instant. The fraction $(1 - H)$ of people without houses get no flow utility, but search for a house. Agents in the economy meet each other with Poisson arrival rate α. When a potential buyer meets a potential seller, they trade the house at price p (which they take as given) with probability $\pi = \pi_0 \pi_1$, where π_0 is the buying probability and π_1 is the selling probability.

(a) Write down the (Hamilton-Jacobi) Bellman equations determining V_1, the value of the owner of a house, and V_0, the value of an agent searching to buy a house. What is the welfare gain of owning a house $V_1 - V_0$?

(b) Characterize the optimal buying and selling strategies which determine π_0 and π_1. Assume that agents trade when they are indifferent.

(c) Find the equilibrium price p. How does it depend on the housing supply H?

C. (30 points) Consider a two period endowment economy with a continuum of individuals of two types. The total population size is 1 and each type has measure $\frac{1}{2}$. Endowments are i.i.d. over time and take on the values 0 or 1 in each period with equal probability of $\frac{1}{2}$. Thus the aggregate endowment is constant at $\frac{1}{2}$ in each period, and the relevant state of nature $s_t \in \{0, 1\}, \quad t = 1, 2$ is an i.i.d. random variable determining which type receives the high endowment of 1 in period t. Both types $i = 1, 2$ of agents have common preferences over consumption c_t^i at the two dates:

$$U(c_1^i, c_2^i) = -E\left[(c_1^i - 1)^2 + \beta(c_2^i - 1)^2\right],$$

where $0 < \beta < 1$. Assume that $0 \leq c_t^i \leq 1$, so that utility is increasing.

Consider efficient allocations with limited commitment. Agents have access to a full set of state-contingent securities, but that they cannot commit to deliver on them. After endowments are realized, agents have the option to default on their promises and revert to autarky for the remainder of time.

(a) If agents can default on their promises at dates 1 and 2, show that no risk-sharing is possible, and autarky is the only equilibrium.

(b) Now assume that agents can default in period 1 on obligations, but they are committed to fulfill their promises in period 2. Thus the allocations are only subject to the participation constraints at date 1, which require that conditional on the date 1 endowment realization the expected discounted utility under the allocation must be weakly better than under autarky. Find the symmetric stochastic steady state (SSSS) equilibria, where an agent consumes $c_t^i = x$ if $e_t^i = 1$ and $c_t^i = 1 - x$ if $e_t^i = 0$.

(c) Compute the equilibrium risk-free rate in the SSSS with partial risk sharing. How do the interest rates and the allocations compare to what they would be under complete markets. What happens when $\beta \to 1$?

Question 2 (100 total points)

A. (50 points) Consider the following growth model. There is a representative household whose utility function is:

$$\sum_{t=0}^{\infty} \beta^t \log(C_t)$$

where C_t is consumption and $\beta \in (0,1)$. The resource constraint is given by:

$$C_t + K_{t+1} = AK_t, \quad \forall t \geq 0$$
$$K_0 = \bar{K}_0$$

where K_t is capital, $A > 1/\beta$, and $\bar{K}_0 > 0$.

The government does not consume any goods, but it has some initial debt outstanding, $\bar{B}_0 > 0$, which it needs to deal with. It can raise revenue by taxing capital income at a constant rate τ and by issuing debt B_t. Let r_t denote the rental rate of capital. Tax revenue in period t is then $\tau r_t K_t$.

(i) Define a competitive equilibrium for this model.

(ii) Calculate a competitive equilibrium. That is, derive an expression for all equilibrium variables as functions of the model primitives β, A, $\bar{K}_0$, and $\bar{B}_0$. [Hint: The recommended approach here is to first make a good guess and then verify it. A good guess here is that equilibrium consumption C_t and investment K_{t+1} are constant multiples of the current capital stock K_t.]

(iii) There is empirical evidence which suggests that countries that start out with high government debt to GDP ratios tend to grow slower than others. Is the model consistent with this fact?

B. (50 points) Consider the following stochastic endowment economy. There are two periods, $t = 1, 2$, and two households, $i = 1, 2$. Let y_t^i denote household i's endowment in period t. Endowments in period $t = 1$ are known with certainty to be $y_1^1 = 1$ and $y_1^2 = 1$. Endowments in period $t = 2$ are uncertain and depend on the future state of the world $s \in \{1, 2\}$, where $\Pr(s = 1) = \Pr(s = 2) = 1/2$. Both households have the same utility function:

$$(c_1^i)^{1/2} + \mathbb{E}\left[(c_2^i)^{1/2}\right]$$

where c_t^i is household i's consumption in period t and $\mathbb{E}$ is an expectations operator. Households are allowed to trade risk free bonds only.

(i) Define a competitive equilibrium for this model economy.

(ii) Suppose that the endowments in period $t = 2$ are given by:

$$y_2^1(s=1) = 1, \quad y_2^1(s=2) = 1,$$
$$y_2^2(s=1) = 1, \quad y_2^2(s=2) = 1.$$

Is the equilibrium allocation Pareto optimal?

(iii) Suppose that the endowments in period $t = 2$ are given by:

$$y_2^1(s=1) = 2, \quad y_2^1(s=2) = 0,$$
$$y_2^2(s=1) = 0, \quad y_2^2(s=2) = 2.$$

Is the equilibrium allocation Pareto optimal?