A Framework for Debt-Maturity Management*

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Abstract

We characterize the optimal debt-maturity management problem of a government in a small open economy. The government issues a continuum of finite-maturity bonds in the presence of liquidity frictions. We find that the solution can be decentralized: the optimal issuance of a bond of a given maturity is proportional to the difference between its market price and its domestic valuation, the latter defined as the price computed using the government's discount factor. We show how the steady-state debt distribution decreases with maturity. These results hold when extending the model to incorporate aggregate risk or strategic default.

Keywords: Debt maturity; Debt management; Liquidity costs.

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1 Introduction

How should a government manage its debt maturity structure? This paper presents a new framework to think about maturity management. The framework makes two innovations. First, it puts forth the importance of liquidity frictions, namely the notion that the larger an issuance at a given maturity the lower the price. Liquidity costs have been well documented by the empirical literature, but have received much less attention by normative theory. The second innovation is technical. The curse of dimensionality quickly restricts the number and class of bonds that can be considered in debt-management problems. Quantitative studies typically model bonds that mature exponentially and work with two maturities only.[1] In practice, however, governments issue finite-life bonds in many maturities. The framework here allows us to work with any number of bonds of arbitrary coupon structure.

The goal of the framework is to analyze an optimal debt-maturity design in the presence of liquidity frictions. To this end, we lay out a continuous-time, small open economy. A relatively impatient government chooses to issue or (re-)purchase finite-life bonds within a continuum of maturities. Its financial counterparts are risk-neutral international investors. The government's objective is to smooth consumption. It faces income and interest rate risk and can default when those risks materialize. Liquidity costs emerge because bonds are auctioned to primary dealers that need time to liquidate their bond holdings after an auction. The bond market is segmented across maturities and vintages, in the spirit of Vayanos and Vila (2009). Under these assumptions, the larger the auction, the lower the price. The induced price impact is summarized by a single coefficient that increases with the holding costs of intermediaries, but decreases with the size of order flows.

We characterize the solution to the government's problem and show that the optimal issuance problem can be decentralized. Namely, the problem can be studied as if the government delegates issuances to a continuum of subordinate traders, each in charge of managing a single maturity. Each trader then applies a simple rule to determine how much to issue of his maturity:

$$\frac{\text{issuance at maturity}}{\text{GDP}} = \frac{\text{relative value gap}}{\text{liquidity coefficient}}.$$

This rule states that the optimal issuance of a bond of a given maturity is the ratio of a relative value gap to the liquidity coefficient. The relative value gap of a bond of a given maturity is the difference between the bond price in the secondary market and the domestic valuation, relative to the secondary market price. The domestic valuation is the counterfactual bond price computed using the government's discount factor, which differs from the international short-

[1] This limitation is easily understood with a simple example. If we want to construct a yearly model where the government issues a single 30-year, zero-coupon bond, we need at least 30 state variables: a 30-year bond becomes a 29-year bond the following year, and a 28-year bond the year after, and so on. By contrast, a bond that matures by 5 percent every year is still a bond that matures by 5 percent the year after its issuance.

term rate.[2] A positive value gap indicates that the trader in the decentralization scheme would otherwise issue as much debt as possible. That desire is contained by the liquidity costs, captured by the liquidity coefficient, which reduce prices in the primary market. We calibrate the liquidity coefficient using data on turnover rates and intermediation spreads.

The paper is built in layers. In the first layer, we study the problem under perfect foresight, in the second layer we add risk, and in the final layer, we add the option to default. Under perfect foresight, the asymptotic dynamics of the model can be obtained analytically. Provided that liquidity costs are above a certain threshold, the model features a steady state. In the steady state, the government issues at all maturities, but issues greater quantities of long-term bonds. It is optimal to issue at all maturities because liquidity costs are convex, namely the price impact increases with the issuance. The steady-state maturity structure is determined by the desire to spread out issuances to minimize the liquidity costs. Nonetheless, this desire does not produce a uniform issuance distribution. This is because long-term bonds have to be rolled over less frequently than short-maturity ones and, hence, the use of the former is preferable in order to minimize rollover costs. Although issuance flows increase with maturity, the outstanding stock of debt decreases with maturity. This decreasing maturity profile for the debt *stock* is an artifact of bonds having a finite life: as long-term bonds mature, they become short-term bonds. Thus, at steady state, the stock of debt at a given maturity is the accumulation of the issuance flows at higher maturities. We calibrate the model for the case of Spain and obtain a debt profile that resembles that in the data.

We use the framework to characterize the maturity distribution as liquidity frictions vanish. Although there is no steady state distribution below a threshold value of the liquidity coefficient, there always exists a well-defined asymptotic debt distribution. This limit-determinacy result contrasts with the case without liquidity costs, in which the maturity profile is indeterminate.

We also study the transitional dynamics after an unexpected shock. Two forces interact with the liquidity costs to shape the dynamics: *consumption smoothing* and *bond-price reaction*. Consumption smoothing is activated when the path of consumption growth changes the domestic discount factor. As a result, domestic valuations are modified. Consumption smoothing lengthens the maturity during downturns: after a shock produces a temporary drop in consumption, the domestic discount rate remains temporarily high while the economy recovers. This reduces domestic valuations, particularly for longer maturities. The optimal rule thus prescribes the issuance of more debt, especially at longer maturities. The economic intuition is that the government issues more debt during recessions to smooth consumption, especially at longer maturities to avoid the liquidity costs associated with the rollover of debt.

[2] The government's discount factor is computed as the solution of a fixed-point problem in the path of consumption. An imputed consumption path maps to a government discount factor. This discount factor generates a path for debt through the issuance rule. Ultimately, the path for debt produces a new consumption path. In the optimal solution, both consumption paths must coincide.

The bond-price reaction force appears when a shock to short-term rates changes bond prices. This force only plays a role in the presence of liquidity costs. Without liquidity costs, the domestic discount factor coincides with the market interest rate, and the government is indifferent between issuing debt at any two maturities. With liquidity costs, however, the government is not indifferent, as rebalancing between different maturities is costly. If a shock temporarily increases short-term rates, thereby reducing market prices—especially for long maturities, the optimal rule dictates, *ceteris paribus,* a decline in debt issuances and the tilting of the maturity distribution toward shorter maturities. The government thus reduces the issuance of those bonds most affected by the temporary fall in prices. The initial shock produces a decline and posterior recovery in consumption, which activates consumption smoothing —a force that partially mitigates bond-price reaction.

The next layer incorporates risk. Because the state variable is the entire maturity distribution, the characterization of an equilibrium with recurrent shocks faces the same computational challenge as incomplete market models with heterogeneous agents and aggregate shocks (as in Krusell and Smith, 1998, and subsequent literature). To provide an analysis of the government's problem with risk that does not rely on numerical approximations, we study an economy where shocks are anticipated, but occur only once. This approach is useful because we can characterize the risky steady state (RSS), defined as the steady state reached when the government expects a shock, but the shock has not yet materialized.[3] Through the analysis of the RSS, we can study how the anticipation of risk shapes the maturity distribution. We show that the government follows the same issuance rule as in the deterministic case. The anticipation of risk introduces an extra term into the domestic valuations: expected valuations after the shock arrival are adjusted by the ratio of post- to pre-shock *marginal utilities*, reflecting an effective "exchange rate" between consumption goods at different states.

Risk introduces a new force that influences the maturity distribution, *insurance*. Insurance shapes the maturity profile in two ways. First, the government tries to build a hedge. In the absence of liquidity costs, the government may *hedge* the changes in consumption due to interest rate shocks by building a portfolio that offsets the impact of the shock. Liquidity costs make a perfect hedge—a hedge that guarantees equal consumption in all states— too costly. Second, the government tries to *self insure*. Self-insurance shows up in the ratio of marginal utilities because a future drop in consumption increases that ratio and, thus, raises valuations. By raising valuations, self-insurance produces a lower stock of debt in the RSS than in the deterministic steady state (DSS). Self insurance-lengthens the average maturity because long-term bonds are less likely to have expired by the time a shock arrives, thus reducing the expected liquidity costs of debt rollover under a negative shock. Although both hedging and self-insurance are

[3]The concept of RSS is equivalent to the one that appears in Coeurdacier et al. (2011). The analysis of the RSS is the *only* tractable solution that does not rely on approximations. Transitional dynamics prior to the shock are not analytically tractable.

present, our calibration shows that self-insurance is a stronger force.

The final layer incorporates strategic default. We solve the Ramsey problem where the government commits to a debt program, but upon the arrival of an income or interest-rate shock, the government can choose to default. Default is costly and thus only exercised if it renders a higher value than repayment. Once again, the optimal issuance rule determines the maturity profile, but default changes bond prices and valuations. Both bond prices and valuations are corrected by a default risk premium. However, domestic valuations are also adjusted by an additional term, which does not appear in the market price of bonds. We dub this term the *revenue-echo effect*. The revenue-echo effect appears because, in the decentralization of the problem, traders anticipate that a marginal rise in their issuances increases default probabilities during the life of the bond. The increase in default risk in future periods echoes back in time through the prices of all bonds that are outstanding during the life of the bond. This fall in bond prices reduces revenue collections from other issuances. All in all, the revenue-echo effect incorporates the spillover of default risk from one bond to the rest of the portfolio. We analyze how the option to default shapes the maturity distribution at the RSS and find that, for our calibration, the revenue-echo effect dominates all the other forces in shaping the maturity distribution.

A similar revenue-echo effect would appear in a version of the model where liquidity costs depend on the outstanding stock of debt, and not only on issuances, as we study here. The solution with default showcases that the techniques are portable to more general environments. Indeed, section 5 explores several variations and applications of the model, namely (i) alternative specifications of liquidity costs, (ii) the case of a government that can only issue debt at a finite number of maturities, and (iii) the issuance of consols instead of finite-life bonds.

Related literature. Maturity management appears in various areas of finance: international, public, and corporate. The framework here captures forces that have been previously identified and uncovers new ones. An advantage of the framework is that it allows for a large number of securities with a realistic coupon structure.

A central feature of the framework is the presence of liquidity costs. There is a large literature that studies both, theoretically and empirically, the sources and magnitudes of liquidity costs.[4] The formulation of liquidity costs in this paper builds on two ideas. As in Vayanos and Vila (2009), markets for bonds of different maturities are segmented. As in Duffie et al. (2005), issuances are intermediated by dealers that face a high discount and need time to reallocate

[4] There is substantial evidence of liquidity costs in different asset classes, as surveyed by Vayanos and Wang (2013b) or Duffie (2010). In the specific case of fixed-income securities, studies that provide evidence suggestive of the presence of liquidity costs are Cammack (1991), Duffee (1996), Spindt and Stolz (1992), Fleming (2002), Green (2004), Pasquariello and Vega (2009), Krishnamurthy and Vissing-Jorgensen (2012), Lou et al. (2013), and Song and Zhu (2018) for US Treasury markets, by Duffie et al. (2003), Naik and Yadav (2003b), Pelizzon et al. (2016) and Breedon (2018) for sovereign bond markets, Edwards et al. (2007) and Feldhutter (2011) for US corporate bond markets, and Green et al. (2007) for US municipal bond markets. Our calibration is based on inventory-depletion evidence in the US Treasury market documented by Fleming and Rosenberg (2008).

assets. In Vayanos and Vila (2009), a finite mass of long-lived investors demand bonds of a specific maturity from intermediaries. Investors trade bonds of all maturities with intermediaries. The price impact in that model depends on the overall outstanding amount of bonds of a specific maturity. In our framework, a large flow of customers contacts dealers, independent of the outstanding amount at a given maturity. Therefore, in our case, the price impact depends on the amount of issuances because this amount affects the time taken by dealers to offload bonds to investors. Our framework can, however, also be used to study maturity management problems without liquidity costs.

International finance stresses several forces that, in our framework, interact with the liquidity costs and shape the maturity structure. One force is *insurance.* A small open economy effectively has access to complete markets when income shocks are correlated with shocks to the yield curve in a way that allows complete asset spanning, as illustrated by Duffie and Huang (1985). With complete spanning, the optimal maturity design is governed by the formation of a bond-portfolio hedge that insulates the country against income and interest rate risk. But if income shocks are uncorrelated with interest rate shocks, the government cannot exploit the maturity of its portfolio to hedge. In that case, the debt dynamics are governed by self-insurance only, as in Chamberlain and Wilson (2000) or Wang et al. (2016). In between these extremes, there are a range of cases with incomplete asset spanning with a role for both hedging and self-insurance. In our framework, liquidity costs inhibit perfect hedges, so self-insurance and partial hedging interplay to shape the debt-maturity profile, independently of the asset structure.

The second force stressed by international finance is *incentives*. When the government has the option to default, it should take into consideration how current issuance affects the incentives to default in the future and how future default affects current prices. This feature was first introduced in a sovereign debt model by Eaton and Gersovitz (1981), and a large literature has followed.[5] The effect of incentives in our framework is captured by the revenue-echo effect. The literature has further identified two important channels, which we abstract from, through which incentives shape the optimal maturity design. The first channel is debt dilution. Bulow and Rogoff (1988) identified that, without commitment to a debt program, long-term debt is prone to debt dilution. Debt dilution occurs when the price of outstanding bonds fall as a result of a new issuance. The expectation of debt dilution raises the cost of long-term debt. Debt dilution is a force tilting issuances toward short maturities.[6] The second channel, rollover risk, operates in the opposite direction. Cole and Kehoe (2000) identified that a solvent government faces rollover risk if it cannot refinance a large amount of principal and, as a result, is forced

[5] See for example Arellano (2008) and Aguiar and Gopinath (2006), and Aguiar and Amador (2013) and Tomz and Wright (2013) for recent reviews.

[6] Hatchondo and Martinez (2009), Chatterjee and Eyigungor (2012), and Chatterjee and Eyigungor (2015) study quantitatively the positive and normative properties of a model in which the government borrows by issuing long term bonds, a setup that is prone to debt dilution.

to default. Rollover risk is a force toward spreading debt services through many maturities to avoid a large principal payment at any given point.

The literature on sovereign debt has studied setups in which insurance and incentive forces are both present. For example, Arellano and Ramanarayanan (2012) and Hatchondo et al. (2016) study the interaction between insurance and incentives, when the government has access to both short- and long-term debt. Bianchi et al. (Forthcoming) study this trade-off when the government saves in a risk-free asset. Recently, Aguiar et al. (Forthcoming) prove the remarkable property that, under debt dilution, once the country has issued long-term debt, it should let long-term bonds mature rather than refinance them with short-term debt. Further recent contributions that combine insurance, incentives, and rollover risk include, for example, Aguiar et al. (2017) and Bocola and Dovis (2018).[7]

Our paper abstracts away from debt dilution and rollover risk, but contributes to the aforementioned literature along two dimensions. First, as we explained earlier, most of the models employed by this literature restrict the number and types of bonds issued. Our framework presents a methodology in which debt problems can be analyzed without these restrictions. In light of the previous discussion, restrictions on the number of bonds and coupon structure represent a limitation of the sovereign debt literature. For example, if debt dilution is present, hedging may be very expensive if long-term debt is restricted to only bonds of very long maturities. Similarly, rollover risk can be mitigated if the country can spread out its issuances, and is not restricted to a few bonds. By circumventing these ad-hoc restrictions, we believe our framework is a step toward understanding debt management with realistic debt structures, where the government is allowed to design its debt profile to mitigate these incentive problems.[8] Second, this paper is the first to solve the debt management problem under commitment to a debt program. This case is a natural and useful benchmark. It is natural if we think that an independent agency controls the debt maturity profile or if an international organization has a discipline device on a sovereign. It is also a useful benchmark to obtain an upper bound on the welfare gains if we want to establish the benefits from a fiscal rule.

Maturity choice is also a classical theme in public finance models. Models of taxation with commitment, as in Lucas and Stokey (1983), Angeletos (2002) or Buera and Nicolini (2004), show how a menu of bonds that differ in maturity implements a complete market allocation.[9]

[7]Dovis (Forthcoming) studies the optimal risk sharing agreement when the government lacks commitment and income is not verifiable, and shows that empirical features documented and rationalized by the quantitative literature on sovereign borrowing naturally emerge as constrained-efficient outcomes in this setup.

[8]Two alternative frameworks allow for richer debt structures, Sánchez et al. (2018) and Bocola and Dovis (2018). In Sánchez et al. (2018) the government chooses each period total debt and the number of periods to repay this debt in equal installments. In our formulation, this choice corresponds to flat debt maturity profiles that differ in their average duration. In Bocola and Dovis (2018), instead, the government chooses each period total debt and a rate of decay for future promised payments. In both papers, total debt plus an additional state variable (number of periods or rate of decay) are sufficient to describe debt maturity profiles. By constrast, in our paper, we allow in principle for any debt maturity profile.

[9]Barro (2003) considers a tax-smoothing objective to assess the optimal maturity structure of public debt.

Buera and Nicolini (2004) also show that this implementation results in unrealistic debt flows and positions.[10] In our paper, because of the liquidity costs, portfolios have to be rebalanced very smoothly and this induces less extreme positions. Recently, Debortoli et al. (2017) and Debortoli et al. (2018) study maturity choice in similar environments but when the government lacks commitment. These papers show that the government chooses a maturity structure that is approximately flat. Our framework generates allocations that are tilted toward shorter maturities which, in our calibrated exercise, are quantitatively similar to those of the Spanish government.[11]

Our paper is also close to the literature that studies tax smoothing under incomplete markets following Aiyagari et al. (2002).[12] Faraglia et al. (Forthcoming) introduce a new method to solve an extension of Aiyagari et al. (2002) that allows for several finite-life bonds of different maturities. In contrast, our paper develops an analytical method to characterize the solution. Our method allows for a clear decomposition of the different economic forces that shape the debt distribution in response to shocks. A common aspect with Faraglia et al. (Forthcoming) is the importance of liquidity costs to obtain a realistic debt profile.

On the technical front, the framework employs infinite-dimensional optimization techniques similar to those applied in heterogeneous agent models. Lucas and Moll (2014) study a problem with heterogeneous agents that allocate time between production and technology generation. Nuño and Moll (2018) study a constrained-efficient allocation in a model with heterogeneous agents and incomplete markets. Nuño and Thomas (2018) study optimal monetary policy in a heterogeneous-agent model with incomplete markets and nominal debt. The contribution relative to those studies is that this paper introduces the risky steady-state approach to study the impact of aggregate risk. This feature is novel, and allows for an exact characterization that can be used to analyze models with aggregate risk.

[10]Faraglia et al. (2010) show that introducing habits and capital leads leads to bond positions that are even larger and more volatile.

[11]Maturity choice is also a recurrent theme in corporate finance. Leland and Toft (1996) introduce a stationary debt structure, which has been widely adopted in the literature that studies the optimal capital structure. Recent contributions that study optimal maturity choice are, for example, Chen et al. (2012), He and Milbradt (2016) and Manuelli and Sánchez (2018). Our framework can be adapted to answer questions in public or corporate finance.

[12]Recent examples are Bhandari et al. (2017a) and Bhandari et al. (2017b). The first paper shows how the government chooses debt, asymptotically, to minimize the variability of fiscal shocks, defined as the sum of the return on debt and spending shocks. The second paper, building on the previous results, shows that when the model is calibrated using US returns on debt and primary deficits, the optimal maturity of debt does not involve any short positions.

2 Maturity management with liquidity costs

2.1 Model setup

Time is continuous. The model features two exogenous state variables, $\{y(t), r^*(t)\}_{t\geq 0}$, representing an endowment and a short-term interest rate process, respectively. The vector of exogenous states is $X(t) \equiv [y(t), r^*(t)]$. For any variable, whether endogenous or exogenous, we employ the notation $x_\infty \equiv \lim_{t\to\infty} x(t)$ to refer to its asymptotic value and x_{ss} to refer to its steady state value, if it exists. In the case of the exogenous state variables, we use both subscripts indistinctly. All the partial differential equations (PDE) that we present here have exact solutions which are presented in Table A of Appendix A.

Households. We consider a small open economy. There is a single, freely-traded consumption good. The economy is populated by a representative household with preferences over expenditure paths, $\{c\,(t)\}_{t\geq 0}$, given by

$$\mathcal{V}_0 = \int_0^\infty e^{-\rho t} U\left(c\left(t\right)\right) dt,$$

where $\rho \in (0,1)$, $r^*_{ss} < \rho$, is the discount factor and $U\left(\cdot\right)$ is an increasing and concave utility function.

Government. The economy features a benevolent government that issues bonds to foreign investors on behalf of households. Bonds differ by their time to maturity. The time to maturity of a given bond is denoted by τ. Issuances are chosen from a continuum of maturities, $\tau \in [0, T]$, where T is an exogenous maximum maturity. The maturity of a given bond falls with time, $\frac{\partial \tau}{\partial t} = -1$, and the bond is retired once it matures, $\tau = 0$. At maturity, the bond pays its principal, normalized to one unit of good. Prior to maturity, the bond pays an instantaneous constant coupon δ. As we discuss in section 5, our framework can also accommodate bonds with different coupon rates. However, for simplicity, we focus on the case in which all bonds have the same coupon rate.

The outstanding stock of bonds with a time-to-maturity τ at date t is denoted by $f\left(\tau, t\right)$. We call $\{f\left(\tau, t\right)\}_{\tau\in[0,T]}$ the debt profile at time t. The law of motion of $f\left(\tau, t\right)$ follows a Kolmogorov forward equation (KFE):

$$\frac{\partial f}{\partial t} = \iota\left(\tau, t\right) + \frac{\partial f}{\partial \tau}, \tag{2.1}$$

with boundary condition $f(T, t) = 0$. The intuition behind the equation is that, given τ and t, the change in the quantity of bonds of maturity τ, $\partial f/\partial t$, equals the issuances at that maturity, $\iota\left(\tau, t\right)$, plus the net flow of bonds, $\partial f/\partial \tau$. The latter term captures the aging of the outstanding bonds, i.e., the automatic flow from longer to shorter maturities. Issuances, $\iota\left(\tau, t\right)$, are chosen from a space of functions $\mathcal{I} : [0,T] \times (0,\infty) \to \mathbb{R}$ that meets some technical conditions.[13] When negative, issuances are interpreted as bond purchases or repurchases. The initial stock of $\tau-$maturity debt is given, $f(\tau,0) = f_0(\tau)$. The government's budget constraint is:

$$c(t) = y(t) - f(0,t) + \int_0^T [q(\tau,t,\iota)\,\iota(\tau,t) - \delta f(\tau,t)]\, d\tau, \tag{2.2}$$

where $y(t)$ is the (exogenous) output endowment, with steady state value y_{ss}. Furthermore, $-f(0,t)$ is the principal repayment, $\delta\int_0^T f d\tau$ are total coupon payments from all outstanding bonds, and $\int_0^T q\iota d\tau$ are the funds received from debt issuances at all maturities. Finally, $q(\tau,t,\iota)$ is the issuance price of a bond of maturity τ at date t in the primary market. As we discuss below, the price depends on the total volume of issuances of that maturity τ.

International investors. The government trades bonds with competitive risk-neutral international investors. The issuance price, q, has two components, a frictionless market price and a liquidity cost:

$$q(\tau,t,\iota) = \underbrace{\psi(\tau,t)}_{\text{market price}} - \underbrace{\lambda(\tau,t,\iota)}_{\text{liquidity costs}}. \tag{2.3}$$

The first component, $\psi(\tau,t)$, is the arbitrage-free market price of the domestic bond. This price depends on the path of the international risk-free interest rate, $r^*(t)$. Given this risk-free rate, the market price $\psi(\tau,t)$ has a PDE representation:

$$r^*(t)\,\psi(\tau,t) = \delta + \frac{\partial\psi}{\partial t} - \frac{\partial\psi}{\partial\tau}, \tag{2.4}$$

with a boundary condition $\psi(0,t) = 1$. Unless indicated otherwise, we assume that $\delta = r^*_{ss}$.

Liquidity costs. The second component in the issuance price, $\lambda(\tau,t,\iota)$, represents a *liquidity cost* associated with the issuance (or purchase) of ι bonds of maturity τ at date t. In Appendix B, we present a wholesale-retail model of the bond market, whose solution yields the formulation of the liquidity cost that we employ throughout the paper. This is similar in spirit to models that feature over-the-counter (OTC) frictions, like Duffie et al. (2005). The main virtue of this friction is that it produces a price impact of issuances, which has been extensively documented in asset markets.[14]

In particular, we assume that to issue bonds of maturity τ at date t, the government decides

[13] In particular, $\mathcal{I}$ is the space of functions $g(\tau,t)$ on $[0,T] \times [0,\infty)$ such that $e^{-\rho t}g$ is square Lebesgue-integrable.

[14] See Foucault et al. (2013) for a textbook treatment of liquidity and market micro-structure and Stoll (2003), Madhavan (2000) and Vayanos and Wang (2013a) for reviews. The three main economic forces explored by this literature are: (i) inventory management of financial intermediaries, (ii) adverse selection and, (iii) order processing costs. Seminal inventory problem papers are Ho and Stoll (1983), Huang and Stoll (1997), Grossman and Miller (1988), and Weill (2007). In the case of adverse selection, see Glosten and Milgrom (1985), Kyle (1985) or Easley and O'hara (1987). A general reference of the implications of order processing costs is Foucault et al. (2013). A large empirical literature has documented the presence of price impact. For example, see Madhavan and Smidt (1991), Madhavan and Sofianos (1998), Hendershott and Seasholes (2007) or Naik and Yadav (2003a,b).

to auction $\iota(\tau, t)$ bonds. The participants in that auction are a continuum of investment bankers. Bankers participate in the auction and buy the total bond issuance. This is the wholesale market. Then, bankers offload bonds to international investors in a retail (secondary) market. As in Duffie et al. (2005), bankers have higher costs of capital than investors. In particular, bankers' cost of capital is $r^*(t) + \eta$, where $\eta > 0$ is an exogenous spread. In the retail market, bankers are continuously contacted by investors. The contact flow is μy_{ss} per instant. Each contact results in an infinitesimal bond purchase by investors from a banker's bond inventory. Thus, in an interval Δt, the stock of bonds sold by the banker is $\mu y_{ss} \Delta t$. We assume that bankers extract all the surplus from the international investors.

The key friction that translates into a liquidity cost for the government is that it takes time for bankers to liquidate their bond portfolios. This, together with the fact that bankers have an inventory holding cost due to the cost of capital, implies that the larger the auction the longer the waiting time to resell a bond and, thus, the lower the price the banker is willing to offer in an auction. As auction size vanishes, the opposite occurs, and the price converges to the market price $\psi(\tau, t)$. These properties are common to OTC models, e.g., Duffie et al. (2005). In Appendix B, we present an exact solution to the auction price as a function of the market price and the issuance size. Here we present an approximation, a first-order Taylor expansion around $\iota = 0$, which yields a convenient linear expression for the auction price:

$$q(\iota, \tau, t) \approx \psi(\tau, t) - \frac{1}{2} \underbrace{\frac{\eta}{\mu y_{ss}}}_{\bar{\lambda}} \psi(\tau, t)\iota(\tau, t). \tag{2.5}$$

Thus, the approximate liquidity cost function is $\lambda(\tau, t, \iota) \approx \frac{1}{2}\bar{\lambda}\psi(\tau, t)\iota$. The term $\bar{\lambda}$ is a liquidity cost coefficient which increases with the spread and decreases with the contact flow.

More General Liquidity Costs. An implicit assumption behind this formulation is that there are no congestion externalities: the contact flow is independent of the outstanding debt at a given maturity. We may consider two departures to this case. First, we could allow for a price impact that spills across maturities and, in that case, equation (2.5) would depend on the entire issuance function, $\iota(\cdot, t)$, and not only the issuance at a given maturity. This departure would still feature liquidity costs that depend on issuances. A second departure would be to allow for liquidity costs that depend on the stock of outstanding debt. In principle our methodology can also deal with that situation. In fact, the solution would share a similarity with the case of default analyzed in section 4. We discuss this connection again in section 5.

We now return to the government's problem and make no further reference to the source of the liquidity costs.

Government problem. The government maximizes the utility of the households given by

$$V\left[f\left(\cdot,0\right)\right]=\max_{\{\iota(\tau,t),f(\tau,t),c(t)\}_{t\geq 0,\tau\in[0,T]}}\int_{0}^{\infty}e^{-\rho t}U\left(c\left(t\right)\right)dt, \tag{2.6}$$

subject to the law of motion of debt (2.1), the budget constraint (2.2), the initial condition f_0 and debt prices (2.3). The object V is the *value functional*, which maps the initial debt profile given by f_0, into a real number. It is a functional because the state variable is infinite-dimensional.[15]

2.2 Solution: the debt issuance rule

We can employ infinite-dimensional optimization techniques to solve the government's problem. This section presents a gist of the approach. A full proof is in Appendix C.1. The main idea is to formulate a Lagrangian:

$$\begin{aligned}\mathcal{L}\left[\iota,f\right] &= \int_{0}^{\infty}e^{-\rho t}U\left(y\left(t\right)-f\left(0,t\right)+\int_{0}^{T}\left[q\left(t,\tau,\iota\right)\iota\left(\tau,t\right)-\delta f(\tau,t)\right]d\tau\right)dt\\ &\quad+\int_{0}^{\infty}\int_{0}^{T}e^{-\rho t}j\left(\tau,t\right)\left(-\frac{\partial f}{\partial t}+\iota\left(\tau,t\right)+\frac{\partial f}{\partial\tau}\right)d\tau dt,\end{aligned}$$

where we substitute consumption and bond prices from the objective function using the budget constraint (2.2) and the bond price schedule (2.3). The second line is the law of motion of the debt distribution, to which we attach the Lagrange multiplier $j\left(\tau,t\right)$. The necessary conditions are obtained by a classic variational argument. The idea is that, at the optimum, the optimal issuance and debt paths cannot be improved. A first condition is that no infinitesimal variation around the control ι can produce an increase in the Lagrangian. This implies that:

$$U'\left(c(t)\right)\left(q\left(t,\tau,\iota\right)+\frac{\partial q}{\partial\iota}\iota\left(\tau,t\right)\right)=-j\left(\tau,t\right). \tag{2.7}$$

This necessary condition is intuitive: the issuance of a (τ,t)-bond produces a marginal cost and a marginal benefit, and both margins must be equal at an optimum. The marginal benefit is the marginal utility of the marginal increase in consumption, which equals the average price of that bond, q, plus the price impact of an additional issuance, $\frac{\partial q}{\partial\iota}\iota$. The marginal cost of the issuance is summarized in the Lagrange multiplier, $-j$, which captures the forward-looking information on the bond repayment, as explained next.

A second condition is that no infinitesimal variation over the state f can yield an improvement in value. The solution cannot be improved as long as the Lagrange multipliers j satisfy

[15] An alternative approach to solving for the optimal solutions, proposed by Golosov et al. (2014), Sachs et al. (2016), and Tsyvinski and Werquin (2017), is to analyze the welfare gains of perturbations from (potentially suboptimal) policies observed in practice.

the following PDE:

$$\rho j\left(\tau,t\right)=-U'\left(c\left(t\right)\right)\delta+\frac{\partial j}{\partial t}-\frac{\partial j}{\partial\tau},\ \ \tau\in\left(0,T\right],\tag{2.8}$$

with a boundary condition: $j\left(0,t\right)=-U'\left(c\left(t\right)\right)$ and a transversality condition $\lim_{t\to\infty}e^{-\rho t}j(\tau,t)=0$.

Each Lagrange multiplier is forward-looking because it captures future repayment costs in the form of a continuous-time present-value formula. The first term, $-U'\left(c\left(t\right)\right)\delta$, is a disutility flow associated with the marginal cost of coupon payments. The second and third terms, $\partial j/\partial t$ and $\partial j/\partial\tau$, capture the change in flow utility as time and the maturity of the bond progress. For interpretation purposes, it is convenient to translate the multiplier j from utils into consumption units. This change of units is useful to transform each multiplier into a financial cost. Define a transformed multiplier as $v\left(\tau,t\right)\equiv-j\left(\tau,t\right)/U'\left(c\left(t\right)\right)$. We refer to v as the *domestic valuation* of a (τ,t)-bond. Aided with this definition, we re-express the first-order conditions (2.7) and (2.8) as

$$\frac{\partial q}{\partial\iota}\iota\left(\tau,t\right)+q\left(t,\tau,\iota\right)=v\left(\tau,t\right),\tag{2.9}$$

and the PDE,

$$r\left(t\right)v\left(\tau,t\right)=\delta+\frac{\partial v}{\partial t}-\frac{\partial v}{\partial\tau},\ \text{if}\ \tau\in\left(0,T\right],\tag{2.10}$$

with terminal condition $v\left(0,t\right)=1$. The rate $r\left(t\right)$ is defined as

$$r\left(t\right)\equiv\rho-\frac{U''\left(c\left(t\right)\right)c(t)}{U'\left(c\left(t\right)\right)}\frac{\dot{c}(t)}{c(t)}.\tag{2.11}$$

Different from $r^{*}\left(t\right)$, the rate $r\left(t\right)$ is the infinitesimal *domestic discount factor*. Under Constant Relative Risk Aversion (CRRA) utility, $U(c)=\frac{c^{1-\sigma}-1}{1-\sigma}$, the domestic discount factor satisfies the classical formula $r(t)=\rho+\sigma\dot{c}(t)/c(t)$. There is a remarkable connection between the domestic valuation (2.10) and the market-price equations (2.4): both are net present values of the cash flows associated with each bond, the only difference between them is the interest rate used to discount the flows. The market price of the bond is computed discounting at r^{*}, whereas the domestic valuation uses r. The optimal issuances given by (2.9) depend on the spread between the two valuations. The next proposition summarizes the discussion and provides a full characterization of the solution:

Proposition 1. *(Necessary conditions) If a solution* $\{c(t),\iota\left(\tau,t\right),f\left(\tau,t\right)\}_{t\geq0}$ *to (2.6) exists then domestic valuations satisfy equation (2.10), optimal issuances* $\iota\left(\tau,t\right)$ *are given by the issuance rule (2.9) and the evolution of the debt distribution can be recovered from the law of motion for debt, (2.1), given the initial condition* f_0. *Finally,* $c(t)$ *and* $r(t)$ *must be consistent with the budget constraint (2.2) and the formula for the internal discount, (2.11).*

Proof. See Appendix C.1. □

There are two noteworthy features of the solution. The first feature is a *decentralization* result: We can interpret the solution as if the government designates a continuum of traders, one for each τ, to give an domestic price to its debt, given the domestic discount factor r. Each trader then issues debt according to the rule in (2.9). The discount factor, of course, must be internally consistent with the consumption path produced by issuances.

The second feature is that issuances are given by a simple debt *issuance rule*. Considering the liquidity cost function (2.5), the optimal-issuance condition (2.9) can be written as:

$$\iota(\tau,t)=\underbrace{\frac{\psi(\tau,t)-v(\tau,t)}{\psi(\tau,t)}}_{\text{value gap}}\cdot\underbrace{\frac{1}{\bar{\lambda}}}_{1/\text{liquidity coefficient}} \tag{2.12}$$

The rule states that the optimal issuance of a (τ,t)-bond should equal the product of the value gap and the inverse liquidity coefficient. This value gap is the difference between the market price, ψ, and the domestic valuation of a bond, v, relative to the market price. When the value gap is positive, a trader wants to issue debt because the market price exceeds the valuation of that debt. A force contains the desire to issue, the liquidity cost, which captures the reduction in price as the issuance becomes larger. This force appears as the coefficient $1/\bar{\lambda}$. The lower $\bar{\lambda}$, the greater the issuance.

The government's problem has a *dual* cost-minimization representation. The dual problem minimizes the net present value of financial expenses, given the discount $r\left(t\right)$ constructed out of a desired consumption path $c(t)$. Formally, the dual problem is:

$$\min_{\{\iota(\tau,t)\}_{t\geq 0,\tau\in[0,T]}}\int_0^\infty e^{-\int_0^t r(s)ds}\left(f\left(0,t\right)+\int_0^T\delta f\left(\tau,t\right)d\tau-\int_0^T q(\tau,t,\iota)\iota(\tau,t)d\tau\right)dt \tag{2.13}$$

where $r(t)$ is given by (2.11), the minimization is subject to the law of motion of debt (2.1) and the initial condition f_0, and debt prices satisfy (2.3). The expression in parentheses is the net flow of financial receipts. This dual problem is consistent with a debt-management office mandate to minimize the financial expenses of a given expenditure path. The proof is in Appendix C.2.

2.3 Asymptotic behavior

The long-run behavior of the solution can be characterized analytically, as shown in Appendix C.3. In some instances, the solution reaches a steady state and in others the solution converges asymptotically to zero consumption. Whether there is a well-defined steady state with positive consumption depends on the value of $\bar{\lambda}$. In particular, we obtain an expression for the *threshold* value $\bar{\lambda}_o$. A steady state exists if and only if $\bar{\lambda}>\bar{\lambda}_o$. If $\bar{\lambda}\leq\bar{\lambda}_o$, there is no steady state;

consumption decreases asymptotically at the exponential rate $r^*_{ss} - \rho$ and $r(t)$ converges to a limit value $r_\infty(\bar{\lambda})$. The asymptotic discount factor $r_\infty(\bar{\lambda})$ is increasing and continuous in $\bar{\lambda}$ with bounds $r_\infty(\bar{\lambda}_o) = \rho$ and $r_\infty(0) = r^*_{ss}$.

In the case where $\bar{\lambda} > \bar{\lambda}_o$, the solution renders an analytic expression for the steady state, with market prices and valuations:

$$\psi_{ss}(\tau) = \delta \frac{1 - e^{-r^*_{ss}\tau}}{r^*_{ss}} + e^{-r^*_{ss}\tau} \text{ and } v_{ss}(\tau) = \delta \frac{1 - e^{-\rho\tau}}{\rho} + e^{-\rho\tau}. \tag{2.14}$$

Assuming that $\delta = r^*_{ss}$, all bonds are issued at par, $\psi_{ss}(\tau) = 1$, and $v_{ss}(\tau) = r^*_{ss}\frac{1-e^{-\rho\tau}}{\rho} + e^{-\rho\tau}$. Issuances at steady state, $\iota_{ss}(\tau)$, follow from (2.12), and the outstanding debt satisfies

$$f_{ss}(\tau) = \int_\tau^T \iota_{ss}(s) ds, \tag{2.15}$$

so their expressions are given by:

$$\iota_{ss}(\tau) = \frac{\rho - r^*_{ss}}{\rho\bar{\lambda}} \left(1 - e^{-\rho\tau}\right), \text{ and } f_{ss}(\tau) = \frac{\rho - r^*_{ss}}{\rho\bar{\lambda}} \int_\tau^T \left(1 - e^{-\rho s}\right) ds. \tag{2.16}$$

We can investigate these expressions to learn about the forces that govern the steady state debt profile. In a deterministic environment, the entire maturity structure in the steady state is determined by the desire to spread out issuances to minimize the liquidity costs. Nonetheless, this does not produce a uniform issuance distribution as long-maturity bonds have to be rolled over less frequently than short-maturity ones, and hence the use of the former is preferable in order to minimize rollover costs. There is, therefore, a tension between issuing debt at long maturities to minimize rollover, and spreading out debt. The solution is that, in the steady state, *issuances are increasing in maturit*y, which can be verified through the derivative of the issuance rule with respect to maturity:

$$\frac{\partial\iota_{ss}}{\partial\tau} = \frac{\rho - r^*_{ss}}{\bar{\lambda}} e^{-\rho\tau} > 0.$$

There is no issuance at the shortest maturity, $\iota_{ss}(0) = 0$, because domestic valuations coincide with market prices. Differences in valuations are higher for longer maturities, as the government discounts future cash flows at a rate ρ greater than r^*_{ss}, whereas the market price of all bonds is constant and equal to 1. This means that, at the margin, the government is willing to receive a lower price on a long-maturity bond, because it requires less frequent retrading.

By contrast to the maturity distribution of issuances, that of debt is *decreasing in maturity*. The reason is simple: because debt is the integral of issuances (see equation 2.15), it should be decreasing in maturity as long as issuances are positive, which is guaranteed by the fact that the government is more impatient than foreign investors:

$$\frac{\partial f_{ss}}{\partial \tau} = -\iota_{ss}(\tau) < 0.$$

Ceteris paribus, as we increase the spread $(\rho - r_{ss}^*)$ the maturity profile shifts toward longer maturities and the overall stock of debt increases. The liquidity cost coefficient decreases issuances in equal proportion.

The analysis above is also useful to discuss what would happen if we allowed the government to issue at any maturity, $T \to \infty$. Notice first how any comparison across economies with different values of the maximum available maturity, T, should be done via a normalization to avoid the artificial disappearance of the liquidity costs as the maximum maturity increases. In particular, we must keep the aggregate contact flow constant, i.e., keep a constant $\int_0^T \mu(T)\, d\tau$ as we increase T, which implies $\mu(T) = \mu(1)/T$. As a result of this normalization, the liquidity coefficient depends linearly on the maximum maturity, $\bar{\lambda}(T) = \bar{\lambda}(1)T$. If we take the limit as $T \to \infty$ in the steady state equation (2.16), we obtain a flat debt profile

$$\lim_{T\to\infty} f_{ss}(\tau; T) = \lim_{T\to\infty} \frac{\rho - r_{ss}^*}{\rho\bar{\lambda}(1)T}\left(T - \tau + \frac{e^{-\rho T} - e^{-\rho\tau}}{\rho}\right) = \frac{\rho - r_{ss}^*}{\rho\bar{\lambda}(1)},$$

for all τ. As a flat debt profile implies an infinite expenditure in coupon repayments in the budget constraint (2.2) and thus an infinitely negative consumption, we may conclude that no steady state exists and that consumption decreases asymptotically toward zero. Asymptotic individual issuances, $\iota_\infty(\tau)$, also converge to zero as T increases. Asymptotic issuances are increasing in maturity but they progressively converge toward a flat structure.[16] Figure H.1 in Appendix H verifies these results for the particular calibration described below. The figure displays the asymptotic values of consumption, discount factor, and the issuance and debt distributions for different values of T ranging from 20 to 1000 years.

2.4 The cases without liquidity costs and vanishing liquidity costs

Proposition 1 characterizes the optimal debt profile in the presence of liquidity costs. Here we characterize the solution without liquidity costs and at the limit when liquidity costs vanish. As expected, in the absence of liquidity costs, the maturity profile is indeterminate. In contrast, the limit solution as liquidity costs vanish features an uniquely determined debt profile.

[16] These two results regarding issuance are a direct consequence of the fact, explained in Appendix C.3, that the limit discount factor is $r_{ss}^* \le r_\infty(\bar{\lambda}) \le \rho$ and hence $\lim_{T\to\infty} \iota_\infty(\tau; T) = \frac{r_\infty(\bar{\lambda}(T)) - r_{ss}^*}{r_\infty(\bar{\lambda}(T))\bar{\lambda}(T)}\left(1 - e^{-r_\infty(\bar{\lambda}(T))\tau}\right) = 0$ and $\frac{\partial \iota_\infty}{\partial \tau} = \frac{r_\infty(\bar{\lambda}(T)) - r_{ss}^*}{\bar{\lambda}(T)} e^{-\rho\tau} > 0$ with $\lim_{T\to\infty} \frac{\partial \iota_\infty}{\partial \tau} = 0$. In the limit, the domestic discount converges to the risk-free rate, $\lim_{T\to\infty} r_\infty(\bar{\lambda}) = r_{ss}^*$, to ensure that the asymptotic debt distribution is zero, $\lim_{T\to\infty} f_\infty(\tau; T) = \frac{r_\infty(\bar{\lambda}) - r_{ss}^*}{r_\infty(\bar{\lambda})\bar{\lambda}(1)} = 0$, and hence that there is no infinite expenditure in coupon repayments.

Consider first the case without liquidity costs, i.e., $\bar{\lambda} = 0$. The necessary conditions for a solution are still summarized in Proposition 1. Since the issuance rule (2.9) still holds, any interior solution must satisfy an equality between domestic valuations and prices, $v(\tau, t) = \psi(\tau, t)$ for any τ. This in turn requires an equality between the domestic discount factor and the interest rate, $r^*(t) = r(t)$. This feature of the limit solution should be familiar: domestic discount factors are equal to the interest rate in standard consumption-savings problems. This observation is enough to characterize the solution at the limit. Denote the market value of the government's debt as:

$$B(t) \equiv \int_0^T \psi(\tau, t) f(\tau, t) d\tau. \tag{2.17}$$

Proposition 7 in Appendix C.4 shows that any solution of the government problem with $\bar{\lambda} = 0$ yields the same value as a consumption-savings problem with a single instantaneous bond in amount $B(t)$, a budget constraint $\dot{B}(t) = r^*(t)B(t) - y(t) + c(t)$ and an initial condition given by $B(0)$. Hence, there are infinitely many solutions that satisfy (2.17) for a $B(t)$ that solves the problem with an instantaneous bond. The intuition is simple, given that the yield curve is arbitrage-free and the discount factor coincides with the interest rate, there is no way to structure debt to reduce the cost of servicing debt. All bonds are redundant, but the path of consumption is consistent with $r^*(t) = r(t)$ and an intertemporal budget.

Now consider the limit solution as $\bar{\lambda} \to 0$. In Appendix C.5, Proposition 8, we present a general formula for the limiting distribution. In the particular case where $\delta = r^*_{ss}$, the *limiting issuance distribution* is determinate and equal to:

$$\lim_{\bar{\lambda} \to 0} \iota_\infty(\tau) = \frac{1 - e^{-r^*_{ss}\tau}}{1 - e^{-r^*_{ss}T}} \kappa,$$

where κ is a positive constant that guarantees that the budget constraint is consistent with zero consumption at the limit. The debt profile shares the qualitative features of the case with positive liquidity costs in Proposition 1; debt issuances are also tilted toward long term bonds, but now ρ plays no role. The result shows that the solution correspondence is upper hemi-continuous, but it is not lower hemi-continuous because for any arbitrarily small cost the distribution is determined. This limiting solution can be employed as a s*election device* that determines the maturity structure.

2.5 Calibration

To provide further insights, we calibrate the model to the Spanish economy. The objective of the calibration is to illustrate the ability of the model to generate realistic debt profiles, and to study the qualitative and quantitative responses to unexpected income and interest rate shocks in the next section. Following Aguiar and Gopinath (2006), we set the value of the coefficient of the

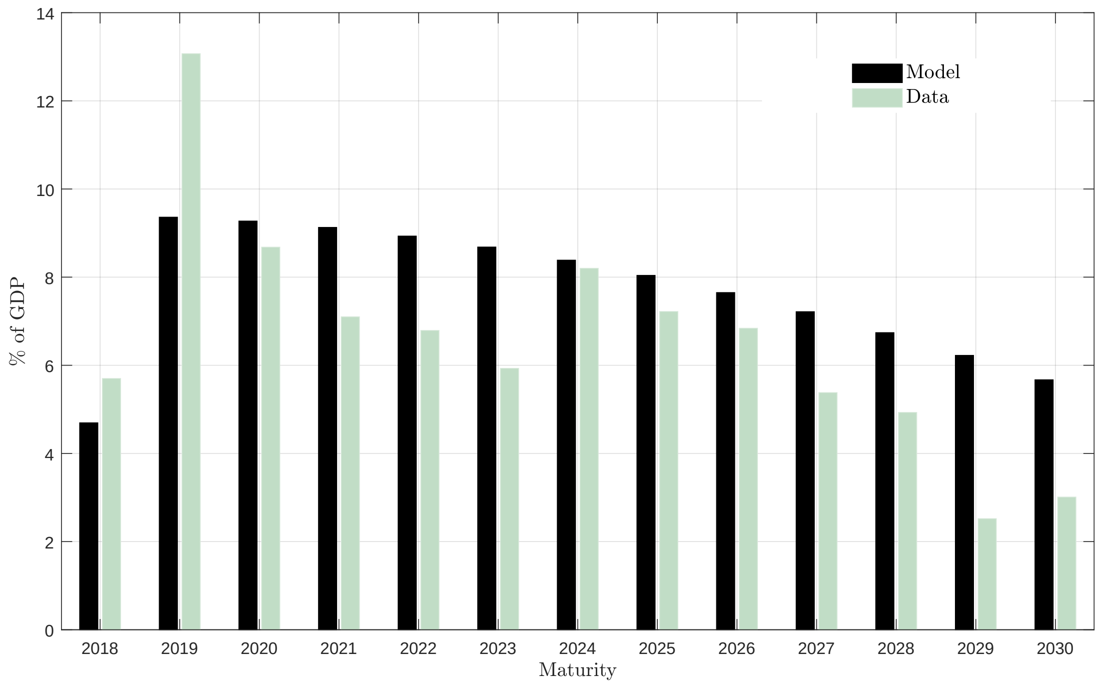

Figure 2.1: Maturity distribution. Data for Spain as of July 2018 and the model in the steady state.

utility function, σ, to 2, and the long-run annual risk-free rate, r^*_{ss}, to 4 percent. We normalize income y_{ss} to 1 and the maximum maturity, T, to 20 years, as roughly 90 percent of Spanish debt has a maturity of less than 20 years. The coupon, δ, is set to 4 percent so the market price equals one at all maturities. Steady state output is normalized to one. In order to calibrate the price impact of bonds, $\bar{\lambda}$, we need to set the values of the arrival rate, μ, and the spread, η, in equation (2.5). In the model, η represents the cost of capital for market makers. We calibrate this spread to 150 basis points. As a reference, we obtain an approximation of the cost of capital of the five biggest US banks in terms of their assets and compute the average spread. More details are provided in Appendix D.

We jointly calibrate the discount factor, ρ, and the arrival rate, μ, to match the average level of Spanish public debt and to replicate the average time that market makers need to exhaust bond inventories. In particular, the calibration tries to match two targets: First, the Spanish government net debt, obtained from the International Monetary Fund *World Economic Outlook* (IMF WEO), was 46 percent of GDP over the period 1985-2016. Second, according to Fleming and Rosenberg (2008), US primary dealers exhaust 60 percent of their inventory one week after they participate in the US Treasury primary market. These two targets imply that, in the steady state,

$$\frac{\mu T}{\int_0^T \iota(\tau)d\tau} = 0.6, \text{ and } \int_0^T f(\tau)d\tau = 0.46.$$

Accordingly, we set the value of μ to 0.0011 and the value of ρ to 0.0416. The implied value of $\bar{\lambda}$ is 7.08. Note that $\frac{\partial q}{\partial \iota}\frac{\iota}{q} \approx \frac{\partial q}{\partial \iota}\frac{\iota}{\psi} = -\frac{1}{2}\bar{\lambda}\iota$. We can compute the elasticity of auction prices with respect to changes in the size of the issue that is produced by this calibration, to get a sense of

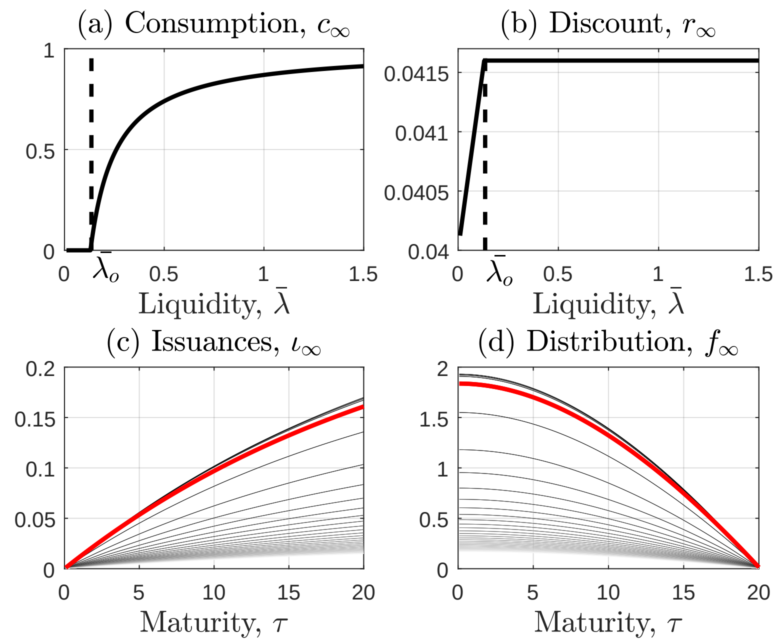

Figure 2.2: Steady-state equilibrium objects as a function of the liquidity cost $(\bar{\lambda})$. Note: The thick red line in panels (c) and (d) indicates the values at the threshold $\bar{\lambda}_o$.

the magnitude of the price impact. The maximum value of issuances is 0.003, for a maturity of 20 years, and hence the elasticity of the auction price equals $-\frac{1}{2} \times 7.08 \times 0.003 = 0.011$. This means that if the government duplicates the size of its steady-state issuance of the 20-year bond, the bond price falls by 1.1 percent at the time of the auction.[17]

Figure 2.1 presents the steady state debt profile generated by the calibrated model alongside the data corresponding to the Spanish debt profile of July 2018. Admittedly, the comparison has some limitations: we compare a steady state object of the model with an empirical counterpart in a particular period.[18] Nevertheless, the figure shows that, despite its simplicity, the model reproduces remarkably well the decreasing maturity profile of Spanish debt. This figure corresponds to outstanding amounts. We discuss the corresponding data for debt issuances in Spain between 2000 and 2018 in Appendix E.

Under the calibration, the model has a proper steady state because the calibration satisfies $\bar{\lambda} > \bar{\lambda}_o$. Figure 2.2 displays steady-state objects as functions of $\bar{\lambda}$. The value of the threshold $\bar{\lambda}_o$ is 0.14, well below the calibrated value of $\bar{\lambda}$, which is 7.08. Panels (a) and (b) show how, for liquidity costs below the threshold, asymptotic consumption c_∞ is zero because no steady state exists. Similarly, as liquidity costs vanish, the discount factor r_∞ converges to the annual risk-

[17] The magnitude of the price impact is roughly similar to other estimates in the literature. For instance, Faraglia et al. (Forthcoming) consider average auction costs of 0.0028 for 10-year, 0.0026 for 9-year and 0.000284 for 1-year bonds based on the empirical findings for the US market of Lou et al. (2013). In our model, calibrated for Spain, these costs amount to 0.0065, 0.0060 and 0.00078, respectively.

[18] Notwithstanding, we have repeated the analysis for other periods with similar results. Results are available upon request.

free rate of 4 percent. The thick red line in panels (c) and (d) indicates the values at the threshold $\bar{\lambda}_o$. The black lines above it are the values for $\bar{\lambda} \leq \bar{\lambda}_o$. The limiting distribution is approached as $\bar{\lambda}$ goes to zero and it is relatively similar to the distribution at the threshold. This implies that, even in the case of liquidity cost values such that no steady state exists, the asymptotic debt and issuance profiles are approximately the same as those at the threshold $\bar{\lambda} = \bar{\lambda}_o$.

2.6 Maturity management with unexpected shocks

The computation of the transitional dynamics that satisfy the analytic expressions of the solution in Proposition 1 requires a numerical algorithm. The idea follows directly from Proposition 1: Given a guess for $\{c(t)\}$, we obtain the domestic discount factor $\{r(t)\}$ through equation (2.11). This discount factor produces valuations $\{v(\tau, t)\}$ according to (2.10), which, in turn, determine the issuances $\{\iota(\tau, t)\}$ through the optimal issuance rule (2.9). Issuances produce a path of debt profiles $\{f(\tau, t)\}$ obtained from the law of motion of debt, (2.1). Given these debt profiles and the corresponding issuance policies, the budget constraint, equation (2.2), determines a new consumption path. The transition to the steady state is a fixed point problem in $\{c(t)\}$, where the guess and resulting consumption paths should coincide. We present the details of this numerical algorithm in Appendix F.

We now study transitions after unexpected shocks to income or to the short-term interest rate. Transitions are initiated from a steady state debt profile. In the experiments, we reset either $y(0)$ or $r^*(0)$ to an initial value, and then let the variable revert back to the steady state. Once either variable is reset, the entire path is anticipated. We study the dynamics of the debt profile. These dynamics are governed by two forces: *consumption smoothing* and *bond-price reaction.* Both forces are counterbalanced by the liquidity costs.

Consumption smoothing refers to the management of the debt profile in order to smooth consumption along a transition path. Consumption smoothing is active only when the government is risk averse, i.e., $\sigma > 0$. This force operates even without liquidity costs, but the force has an effect on the debt maturity profile only when liquidity costs are present, $\bar{\lambda} > 0$.[19] This force is summarized by the internal discount factor $r(t) = \rho + \sigma \dot{c}(t)/c(t)$. In fact, when r^* is constant ($\psi = 1$), the value gap equals $1 - v$. In this case consumption growth shapes the maturity profile directly from the solution to v presented in Proposition 1: if a shock produces a growing consumption path, the domestic discount increases and the debt distribution tilts toward longer maturities, as valuations at longer horizons are affected more. The increase in domestic discounts, or equivalently the decline in domestic valuations, also induces an increase in the volume of issuances.

Bond-price reaction is a more subtle force. This force determines how the debt profile

[19]Without liquidity costs, permanent income and the path of interest rates determine the consumption path, but the debt maturity profile is indeterminate, as we discussed above.

changes with bond prices, $\psi(\tau, t)$. It is present even with risk neutrality, i.e., when we shut down consumption smoothing. This force only plays a role in the presence of liquidity costs, $\bar{\lambda} > 0$. Without liquidity costs, the domestic discount factor coincides with the interest rate, $r(t) = r^*(t)$, and since bond prices are arbitrage free, the government is indifferent between issuing at any two maturities. With liquidity costs, even a risk-neutral government is no longer indifferent. The reason is that liquidity costs allow for a gap between $r(t)$ and $r^*(t)$ as rebalancing is now costly. Any gap between these two rates is compounded differently along different time horizons, and thus the government faces different consumption paths depending on its maturity choice. We can again observe how this force operates through the value gap in the simplified case of risk neutrality. In this case consumption smoothing is not active, and valuations are constant and equal to their steady state value v_{ss}. If a shock temporarily increases short-term rates, thereby reducing market prices −specially at long maturities, the optimal issuance rule (2.12) prescribes a decline in debt issuances and a tilting of the maturity distribution toward shorter maturities. In the general case with risk aversion, interest rate shocks produce a race between consumption smoothing and bond-price reaction.

To gather further insights into these two forces, we analyze the responses to unexpected shocks to $y(t)$ and $r^*(t)$ separately. Consider first the response to an income shock, displayed in figure 2.3. In the experiment, income follows $dy(t) = \alpha_y (y_{ss} - y(t))\, dt$, the continuous-time counterpart of an $AR(1)$ process. We set $y(0)$ to a level 5 percent below its steady state value, corresponding to a major recession in Spain.[20] The reversion coefficient, α_y, is set to 0.2 in line with Spanish data. In this case, the only active force is consumption smoothing, as the interest rate remains constant.

Figure 2.3 displays the transition. Panels (a) and (b) show how the fall in income produces a decline in consumption on impact, followed by a recovery. The expected consumption growth produces an increase in the domestic discount on impact, which reverts back to the steady state. Since there is an increase in domestic discounting, domestic valuations decrease, which acts like a reduction in the perceived cost of debt. Given that bond prices are constant, the optimal issuance rule (2.12) dictates an increase in the issuances at all maturities, as displayed in panel (e) and hence in the outstanding amount of total debt (panels c and f), $b(t) \equiv \int_0^T f(\tau, t)\, d\tau$. One noticeable feature is that new issuances are not homogeneously distributed across maturities: they are more concentrated in longer maturities, as expected. This is because long-term domestic valuations are more sensitive to changes in the discount factor. This produces an *increase* in the average debt duration of the portfolio, computed as the average of the Macaulay duration of each individual bond (panel d).

The intuition for this pattern is the following. In response to a negative income shock, the government attempts to smooth consumption by issuing more debt. Because of the liquidity costs, the government tilts issuances toward long-term debt to minimize the liquidity costs as-

[20] See Appendix D for further details on the calibration of the experiments.

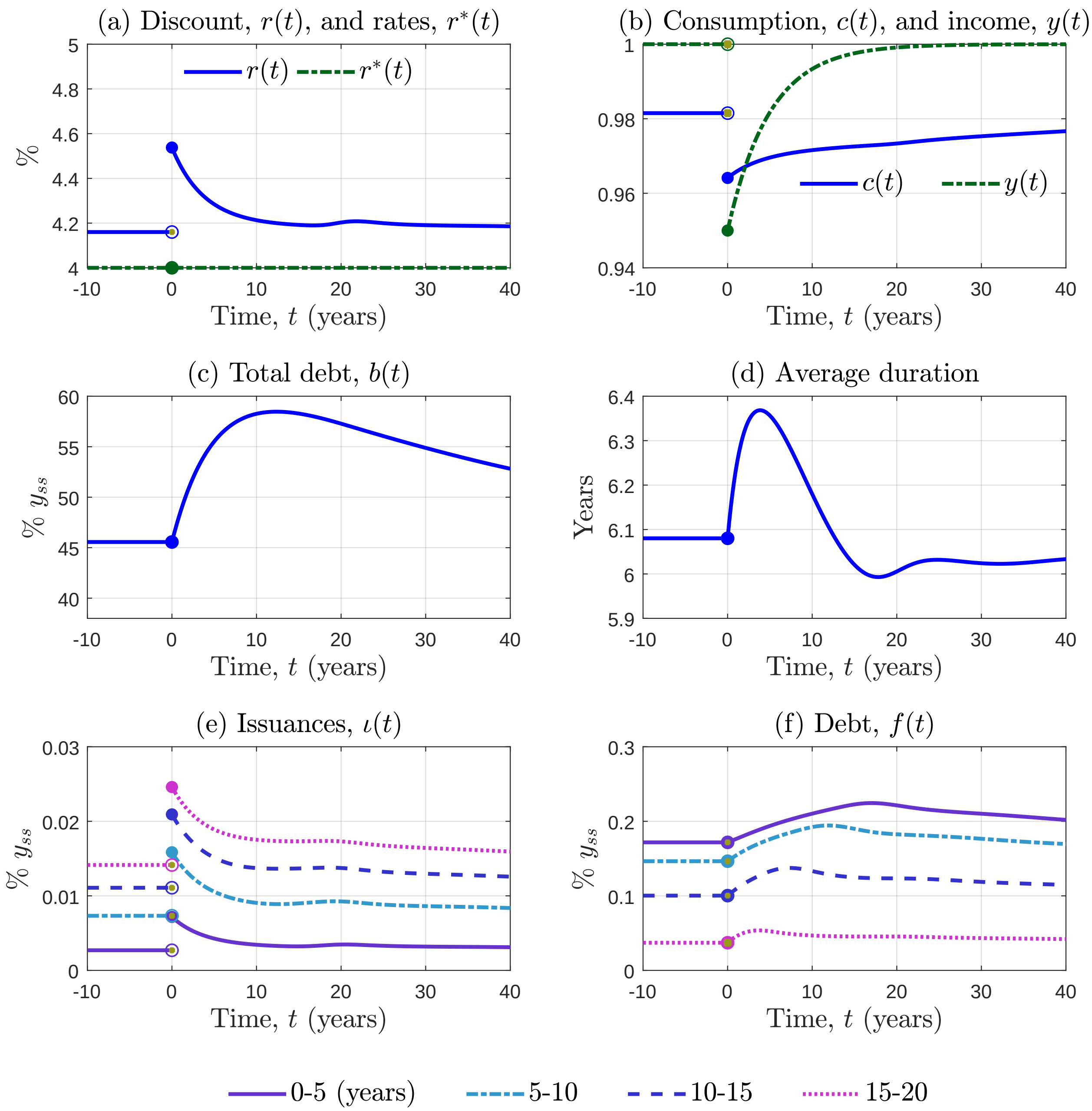

Figure 2.3: Response to an unexpected temporary drop in income.

sociated with debt rollover during the period in which the decline in income, and consumption, is more acute. The desire of the government to delay these rollover costs explains the lengthening in the maturity.

To pry on the mechanism, figure 2.4 displays the optimal transition path when at time 0 it is revealed that a positive income shock lasting one year will hit the economy 20 years ahead (panel b). Prior to the arrival of the shock, the government progressively issues debt that matures when the shock hits. In particular, the first year it issues 20-year bonds, in the second year it issues 19-year bonds, and so on (panel e). As displayed in panels (c) and (d), the government not only steadily decreases the maturity of the issuances, it also increases the amount of debt issued as the date of realization of the shock approaches. The result is a progressive build-up of

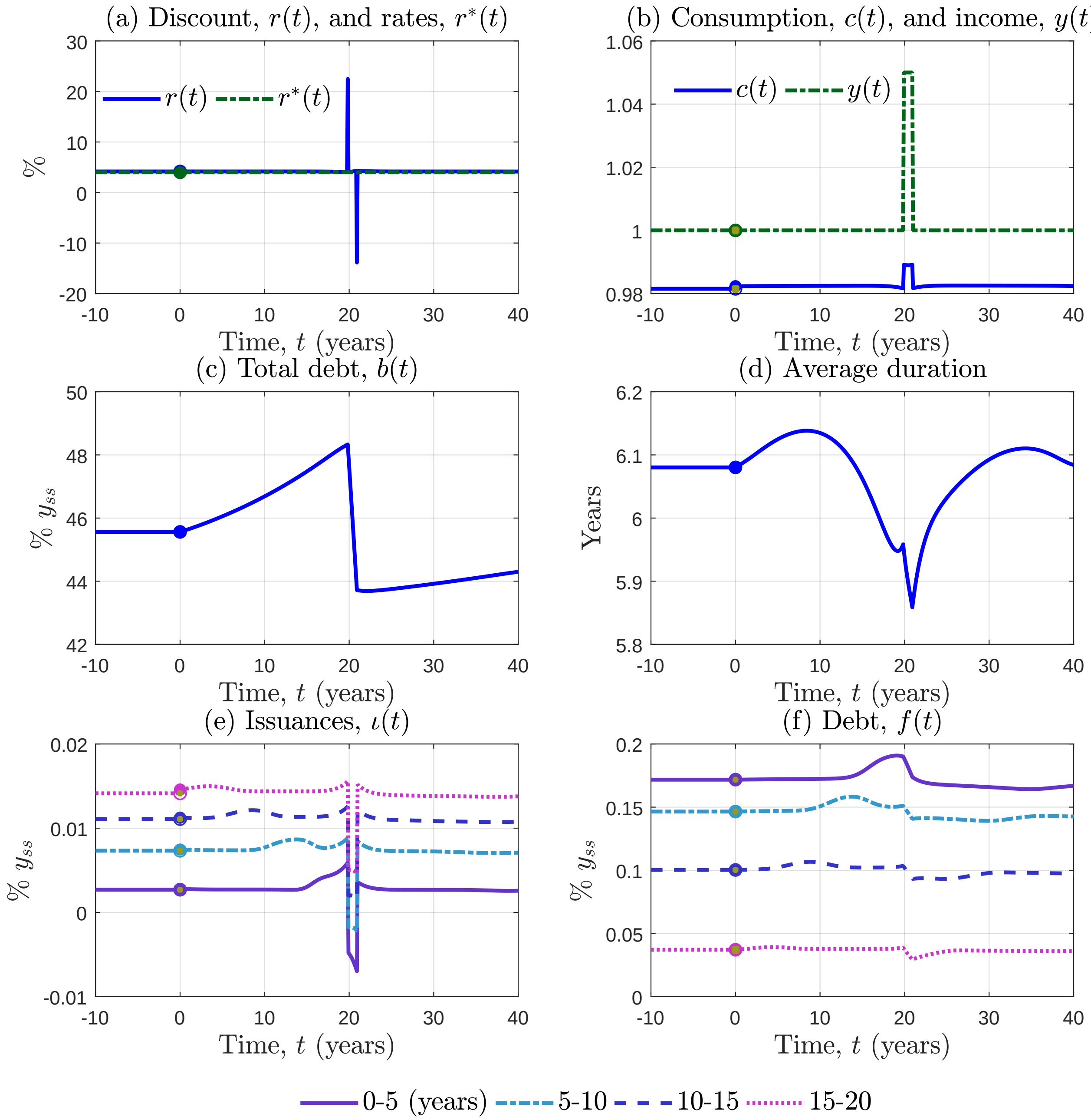

Figure 2.4: Response to an unexpected drop in income 20 years ahead.

debt, which allows a relatively constant consumption path greater than its long-term value. The average duration is longer than its long-term average during the first 14 years, because the government issues bonds with longer maturities than the average (around six years) and declines afterwards until the shock arrival. Once the shock hits, the government uses the extra income to (i) repay the principals of all the debt issued in the pre-shock period, (ii) repurchase part of its debt stock, especially at short-term maturities and (iii) increase temporarily consumption. As in the previous exercise, this strategy minimizes the liquidity costs because the government spreads out bond issuances and avoids any rollover of the extra debt prior to the shock arrival.

Next, consider an unexpected shock to $r^*(t)$, as presented in figure 2.5. We let $dr^*(t) = \alpha_r (r^*_{ss} - r^*(t))\, dt$, with $\alpha_r = 0.2$, which is also taken from the Spanish data. On impact, the

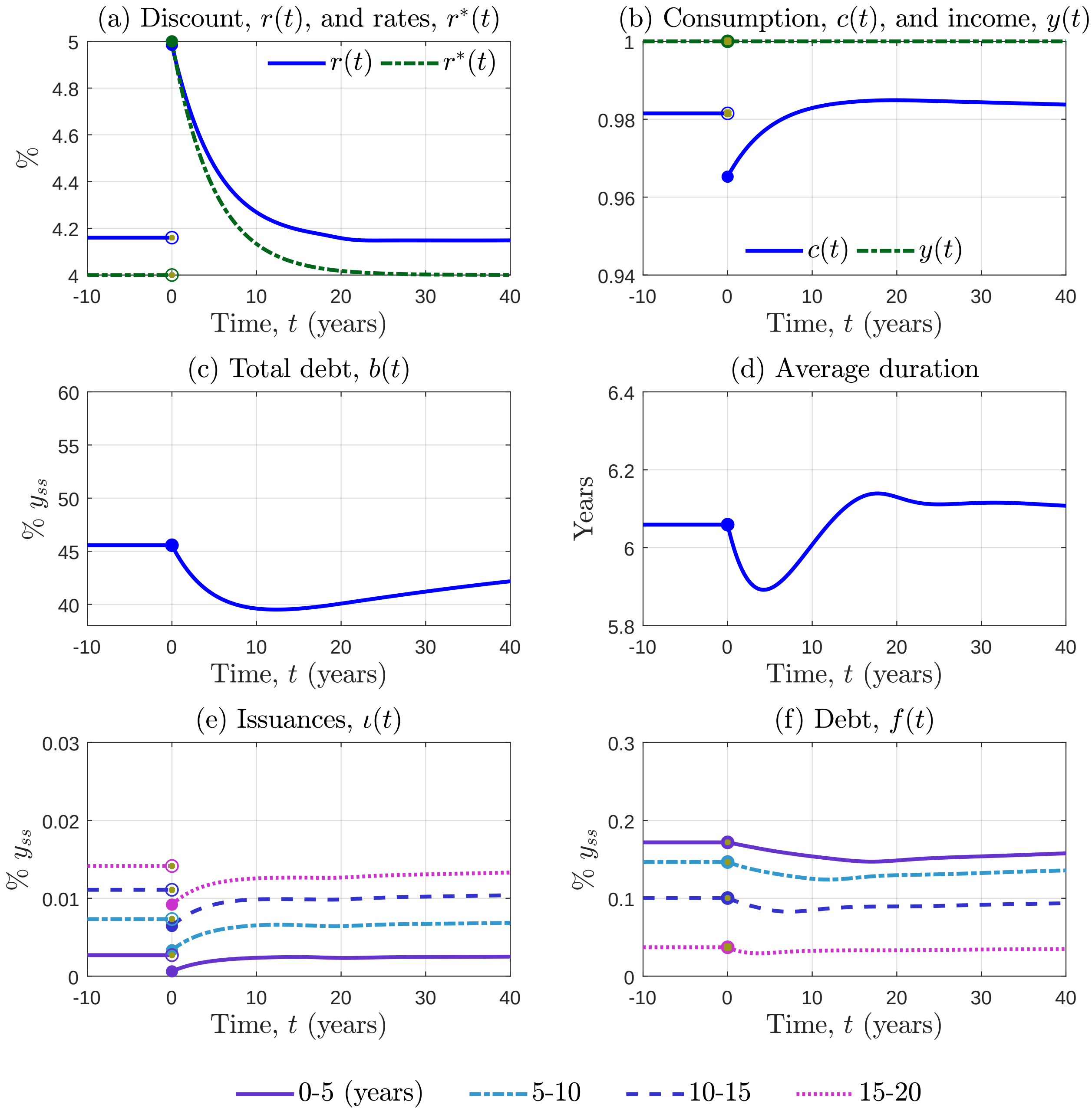

Figure 2.5: Response to an unexpected temporary shock to interest rates.

interest rate increases from 4 percent to 5 percent. The shock produces a consumption drop of the same magnitude as the income shock in figure 2.3, which makes both shocks comparable. In this case, both the bond-price reaction force and consumption smoothing are active.

We can interpret the transition as follows. On impact, the domestic discount jumps to 5 percent, as shown in panel (a) of figure 2.5. This narrows the value gap across all maturities. The initial effect of a narrower value gap is a decrease in all issuances (panel e). This captures the notion that, upon an interest rate increase, the government wants to sacrifice present consumption to mitigate a higher debt burden. A noticeable feature is that the interest rate shock produces an initial *reduction* in the average debt duration. This occurs even though the path of consumption resembles the one of the income shock that produces the opposite prediction regarding debt maturity. The reason why maturity shrinks with the rate shock is that, in this case, the valuation gap decreases more for long-term bonds. The valuation gap is proxied by the interest rate gap, $r(t) - r^*(t)$, which narrows on impact, but widens again as consumption recovers. This tells us that bond-price reaction dominates over consumption smoothing.

The intuition in this case is straightforward. Faced with temporary lower bond prices, the government decides to issue less debt. This reduction is more relevant for long-term bonds, because these are the bonds that experience the larger price declines. This effect is, nevertheless, partially mitigated by consumption smoothing because the initial decline in consumption encourages, as discussed above, the issuance of long-term bonds.

In order to isolate the bond-price reaction force, figure H.2 in Appendix H displays the transitions when the government is risk neutral, i.e, $\sigma = 0$. The effects of figure 2.5 are magnified as consumption smoothing is not active. As figure H.2 shows, once the shock arrives and bond prices fall, the government decides to directly repurchase debt, especially at long maturities, and to resell it when prices go up again. This summarizes well the essence of this force: because liquidity costs make it expensive to rebalance debt across maturities, the government buys (or issues less) debt when it is temporarily cheap in order to resell it (or issue more) when prices recover.

Summing up, the introduction of liquidity costs explains the opposite responses, in terms of the issuance profile, to contractionary shocks. In the case of an income shock, the desire to issue more debt to smooth consumption will necessarily depress bond prices in the primary market, forcing the government to spread issuances to minimize the price impact and tilt them toward long maturities to reduce the rollover during the recession. In the case of an interest rate shock, the previous channel is more than compensated by the change in bond prices in the secondary market, and hence, the government finds it optimal to temporarily reduce issuances in the long part of the yield curve because those are the bonds most affected by the decline in prices.

3 Risk

The previous section analyzed optimal debt-management under perfect foresight. This section presents a characterization that allows for risk. The goal is to understand how the anticipation of shocks affects the shape of the optimal debt profile and the ability to insure.

3.1 The model with risk

Risk is modeled as a single jump at a random date. We introduce notation to distinguish pre- from post-jump variables: for any variable x, we represent its value prior to the shock by a hat, i.e., $\hat{x}(t)$, and use $x(t)$ to express the value after the jump event.

The exogenous state $X(t)$, comprising income, $y(t)$, and the risk-free rate, $r^*(t)$, now features a jump at a random date t^o. Prior to the jump, the state is denoted as $\hat{X}(t)$. The random date t^o is exponentially distributed with a parameter ϕ. Upon the jump, a new value for the state is drawn, $X(t^o) \sim G(\cdot|\hat{X}(t^o))$, where $G(\cdot)$ is the conditional distribution of the exogenous state given its value prior to the shock.[21] We let $\{y(t^o), r^*(t^o)\}$ take values in the compact space $[y_l, y_h] \times [r_l^*, r_h^*]$ and let $G(\cdot|\hat{X}(t^o))$ be a discrete or continuous distribution. We denote the conditional expectation operator under the distribution $G(\cdot; X(t^-))$ by $\mathbb{E}_t^X[\cdot]$.

Once the jump occurs, each element $x(t) \in X(t)$ follows a continuous mean-reverting process:

$$\dot{x}(t) = -\alpha^x (x(t) - x_{ss})$$

where α^x captures the speed of mean reversion to its steady-state value x_{ss}. Obviously, there is risk before the one-time jump, but after the jump the economy becomes the perfect-foresight one of the previous section.

With risk, bond prices satisfy a standard pricing equation with a single jump. The flow value of the bond, δ, plus the expected price appreciation should equal the value of the bond at the market rate $\hat{r}^*(t)\hat{\psi}(\tau, t)$. Thus, the market price is now given by:

$$\hat{r}^*(t)\hat{\psi}(\tau, t) = \delta + \frac{\partial \hat{\psi}}{\partial t} - \frac{\partial \hat{\psi}}{\partial \tau} + \phi \mathbb{E}_t^X \left[\psi(\tau, t, X_t) - \hat{\psi}(\tau, t)\right], \text{ if } t < t^o, \tag{3.1}$$

with the terminal condition $\hat{\psi}(0, t) = 1$. As before, the market price $\psi(\tau, t, X_t)$ denotes the perfect-foresight price of the bond after the shock if the initial state is X_t, which still satisfies (2.4). The price that the government receives is still given by (2.5).

Prior to the shock, the government problem is:

$$\hat{V}\left[\hat{f}(\cdot, 0), X(0)\right] = \max_{\{\iota(\tau,t), f(\tau,t), c(t)\}_{t\geq 0, \tau\in[0,T]}} \mathbb{E}_t \left[\int_0^{t^o} e^{-\rho t} U(\hat{c}(t))\, dt + e^{-\rho t^o} \mathbb{E}_{t^o}^X \left[V\left[\hat{f}(\cdot, t^o), X(t^o)\right]\right]\right], \tag{3.2}$$

where $V\left[\hat{f}(\cdot, t^o), X(t^o)\right]$ is the value given by (2.6). The maximization is subject to the law of motion of debt (2.1), the budget constraint (2.2) and the initial condition $\hat{f}(\cdot, 0) = f_0$, and bond prices are given by (3.1).

3.2 Solution: risk-adjusted valuations

To characterize the problem, we adapt the perfect-foresight solution to allow for risk. The proof of the solution to the problem with risk can be found in Appendix C.6. The approach is

[21] Formally, $X(t)$ is right-continuous. If the jump occurs at t^o, the left-limit $\hat{X}(t^o) \equiv \lim_{s\uparrow t^o} \hat{X}(s)$ jumps to some new $X(t^o) \sim G(\cdot; X(t^o))$.

the same as the one used to prove Proposition 1: We construct a Lagrangian and then apply variational techniques to obtain the first-order conditions. Proposition 2 below is the analogue to Proposition 1.

Proposition 2. *(Necessary conditions) If a solution* $\{c(t), \iota(\tau,t), f(\tau,t)\}_{t\geq 0}$ *to (3.2) exists, it satisfies the same conditions of the perfect-foresight solution (Proposition 1) with valuations prior to the shock now satisfying the PDE*

$$\hat{r}(t)\,\hat{v}(\tau,t) = \delta + \frac{\partial \hat{v}}{\partial t} - \frac{\partial \hat{v}}{\partial \tau} + \phi \mathbb{E}_t^X\left[v(\tau,t)\,\frac{U'(c(t))}{U'(\hat{c}(t))} - \hat{v}(\tau,t)\right],\ \tau \in (0,T], \tag{3.3}$$

with boundary condition $\hat{v}(0,t) = 1$.

Proposition 2 shows how the decentralization scheme of the perfect-foresight problem carries over to the problem with risk: The optimal issuance rule still holds, but applies to the pre-shock valuations, which now are adjusted for risk. After the shock, the transition and asymptotic limits coincide with those of the perfect-foresight problem that takes as its initial condition the debt profile at the time of the shock.

This decentralization shows that the effect of risk is captured exclusively through a correction in domestic valuations. The PDE is similar to the one that corresponds to the riskless case, equation (2.10). The only difference is the last term on the right. This term is akin to the correction in any expected present-value formula that features a jump. In the particular case of domestic valuations, the expression captures that, upon the arrival of the shock, valuations jump from $\hat{v}(\tau,t)$ to $v(\tau,t)$, the latter term being corrected by the ratio of marginal utilities. This ratio of marginal utilities captures an "exchange rate" between states. The domestic valuation after the shock, $v(\tau,t)$, is measured in goods after the arrival of the shock, but the valuation $\hat{v}(\tau,t)$ is expressed in goods prior to the shock. The ratio of marginal utilities tells us how goods are relatively valued in terms of utilities, before *vis-à-vis* after the shock. If a shock produces a drop in consumption, the ratio of marginal utilities associated with that state is greater than one. This correction captures that payments associated with a bond are more costly in states where marginal utility jumps. This extra kick in the valuations affects bonds of different maturities differently, as we illustrate below.

Although Proposition 2 characterizes the solution with risk, its computation involves solving a fixed point problem over a family of debt distributions. The reason for this complexity is that any date t prior to the shock is associated with a jump in consumption from $\hat{c}(t)$ to $c(t)$. The size of the jump is a function of the distribution $f(\cdot,t)$. Through its influence on valuations, the potential consumption jump affects the choice of issuances. Hence, a solution to the transition is a fixed point over a family of debt distributions, requiring a time-varying distribution prior to the shock arrival consistent with the post-shock consumption path. This complexity renders the numerical solution to the problem unfeasible, and we are forced to employ only approximate solutions. If instead of a single jump, we were to consider multiple jumps, the complexity would escalate even further. An analogous challenge appears in incomplete market models with aggregate shocks, which is why the literature uses the bounded-rationality approximation of Krusell and Smith (1998) or equivalent approaches.

Our approach to analyze the influence of risk is to study the risky steady state (RSS) of the problem. In our context, the RSS is defined as the asymptotic limit of variables prior to the realization of the shock. In other words, the RSS is the state to which the solution converges when shocks are expected to arrive but have not yet materialized. We obtain an explicit solution to the valuations in the RSS:

$$\hat{v}_{rss}(\tau) = e^{-(\hat{r}_{rss}+\phi)\tau} + \int_0^\tau e^{-(\hat{r}_{rss}+\phi)(\tau-s)} \left(\delta + \phi \mathbb{E}^X_{rss}\left[\frac{U'(c(0))}{U'(\hat{c}_{rss})} v(s,0)\right]\right) ds, \text{ for } \tau \in (0,T], \tag{3.4}$$

where $v(\tau,0)$ is the valuation of the perfect-foresight problem with initial condition $\hat{f}_{rss}(\tau)$.

The advantage of the RSS approach is that the RSS can be characterized as a fixed point in the space of consumption policies, an object that is feasible to compute. In fact, the method used to calculate a solution adds only one more unknown to the perfect-foresight algorithm; now we must obtain the RSS consumption. The idea is to propose a guess of c_{rss} and the path $\{c(t)\}$ and then compute the valuations using (3.4) for the RSS and (2.10) for the valuation after the shock. We then obtain issuances from the simple issuance rule (2.12). Given issuances, we compute the debt profile in the RSS using the law of motion of debt (2.1). The budget constraint (2.2) yields a new c_{rss} and a new path $\{c(t)\}$. Naturally, $\{c(t)\}$ is the solution to the perfect-foresight problem where $f(\tau,0) = \hat{f}_{rss}(\tau)$.

We can exploit a decomposition to explain how risk alters the debt distribution from the deterministic steady state (DSS) to the RSS. The main force that shapes the maturity profile under risk is *insurance*. Insurance is captured by the ratio of marginal utilities, $U'(c(0))/U'(\hat{c}_{rss})$, in equation (3.3). Insurance is achieved in two ways: via *self-insurance* or *hedging*. By self insurance we refer to the fact that, in some cases, the government may decide to reduce its RSS debt stock to minimize the fall in consumption after a negative income or interest rate shock. Moreover, in the case of an interest rate shock, the government might, in principle, hedge the fall in consumption by building a portfolio that offsets the impact of the shock. In Appendix G.1 we analyze the particular case without liquidity costs and show how, in the case of interest rate shocks, the government may perfectly hedge, whereas in the case of income shocks, it is forced to self-insure.[22]

[22] In the case of interest rate shocks, we derive a generalization of the conditions that guarantee market completion discussed in Duffie and Huang (1985), Angeletos (2002) or Buera and Nicolini (2004).

3.3 The risky steady state

We now return to our calibration to analyze how quantitatively important the forces that risk introduces are in shaping the maturity profile. To do so, we compute the RSS and analyze post-shock transitions. We compare these dynamics with those following unexpected shocks when the economy is at the deterministic steady state. Naturally, in both cases the economy converges to the DSS as time progresses. We calibrate the shock intensity ϕ to 0.02. As explained in Appendix D, the value of the intensity is obtained by computing the cross-country frequency of a yearly output decline greater than or equal to 5 percent. We maintain the rest of the calibration.

The comparisons are reported in figures 3.1 and 3.2 where solid lines denoted by DSS are the dynamics of the perfect-foresight section and dashed lines denoted by RSS describe the dynamics when the shock is expected.

Figure 3.1 reports responses to a shock that produces a 5 percent income drop. Compared to the DSS, the total debt stock in the RSS prior to the shock (panels a and c) is smaller, 35 percent of GDP compared to 45 percent. The reason for this decline is self-insurance. Since rates do not move, both hedging and bond-price reaction are shut down. Thus, the decline in the stock of debt is a race between two forces, ex-post consumption smoothing and ex-ante self-insurance. These forces are encoded in the valuations. Recall from the RSS valuation formula (3.4) that two terms influence RSS valuations, namely the ratio of marginal utilities and post-shock valuations. Panel (f) displays the valuation immediately after the arrival of the shock, $v(\tau, 0)$. This value is lower than the DSS valuations, $v_{ss}(\tau)$, reflecting that once the shock hits, $r(t)$ will increase. Despite $v(\tau, 0)$ being lower than $v_{ss}(\tau)$, the RSS valuations are higher, i.e., $v_{rss}(\tau) > v_{ss}(\tau) > v(\tau, 0)$, and this mechanically occurs because of the larger ratio of marginal utilities $\frac{U'(c(0))}{U'(\hat{c}_{rss})} = \left(\frac{0.986}{0.971}\right)^2 = 1.03$. This tells us that self-insurance and consumption smoothing operate in opposite directions, with self-insurance as the dominant force, thereby producing a lower stock of debt.

Another feature is that duration is slightly higher in the RSS (panel d). This can be entirely explained by the change in the concavity of the valuations in the RSS, $\hat{v}_{rss}(\tau)$, with respect to valuations in the DSS, both displayed in panel (f). As explained in section 4.3, once the shock hits, consumption smoothing extends the average duration to reduce rollover costs. This is captured by the shape of $v(\tau, 0)$, which is inherited by $\hat{v}_{rss}(\tau)$. The intuition is the following. The government slightly tilts the debt distribution toward long maturities to reduce rollover costs in the event of the shock arrival.

Next, consider the case of an interest rate shock. The stock of debt is 30 percent of GDP in the RSS, a smaller value than the 45 percent in the deterministic steady state, as displayed in panel (c) of figure 3.2. The reduction is seen at all maturities (panel a). Again, we can dissect which forces drive these results by examining the effects on valuations and prices. As with the income shock, consumption smoothing operates in the opposite direction of self-insurance, but

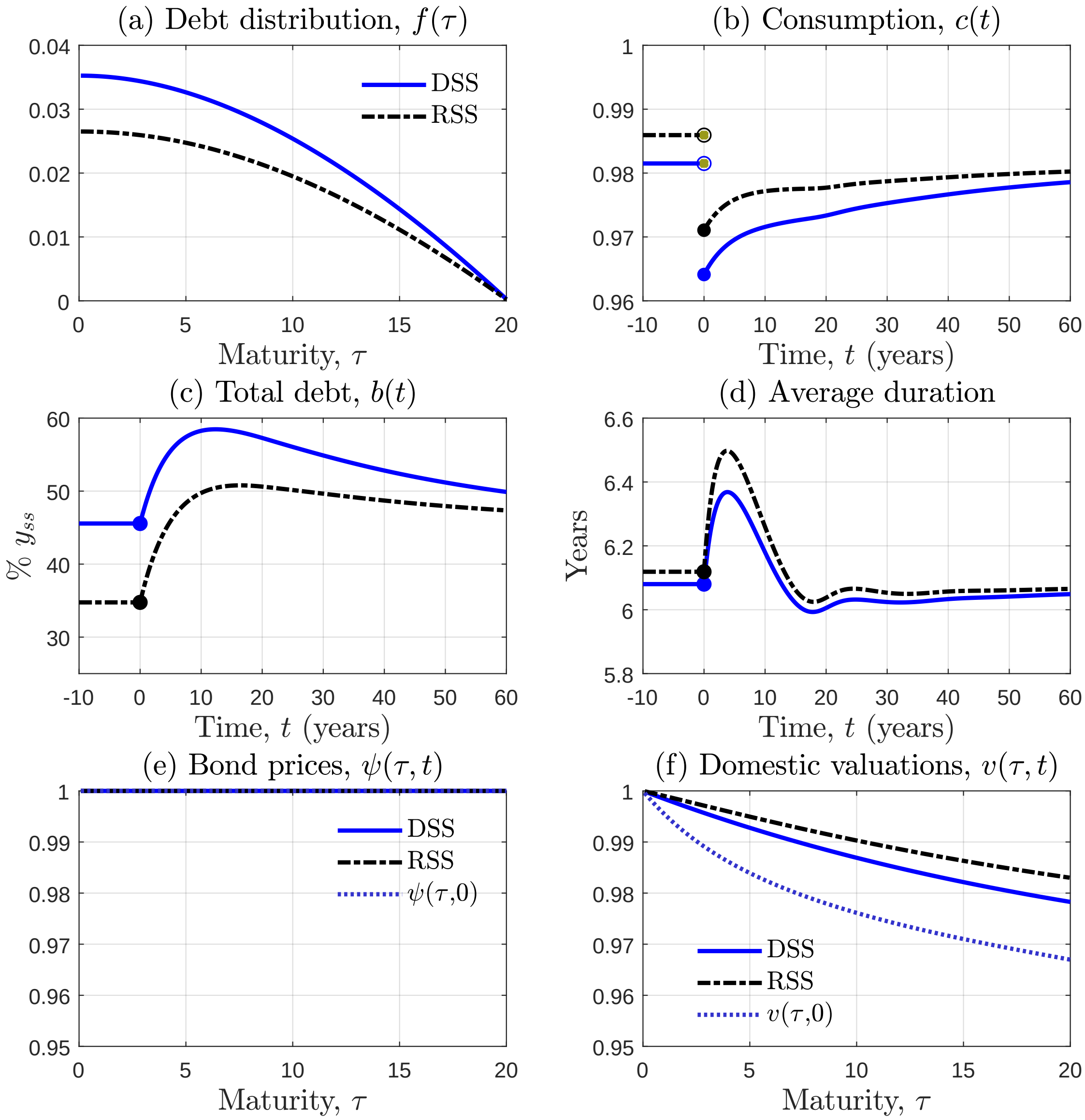

Figure 3.1: Response to a shock to income. RSS is model starting at the risky steady state and DSS at the deterministic steady state.

in this case both forces cancel each other. As shown in panel (f), domestic valuations are very similar between the DSS and the RSS.

In the case of an interest rate shock, hedging might also play a role in shaping the maturity profile. We can understand the role of hedging by comparing the RSS solution to the counterfactual RSS of a risk-neutral government, $\sigma = 0$. This comparison is reported in figure H.3 of the Appendix H. The figure reveals that the overall stock of debt is not modified substantially. However, we do observe some hedging. As discussed in the case without liquidity costs in Appendix G.1, the correct hedge to a negative interest-rate shock is to hold more long-term debt and less short-term debt. We observe that pattern, but the effect is small. This suggests that liquidity costs introduce both a cost to set up the hedge ex-ante and a cost to unwind the position. In our calibration, these costs are large and practically mute the hedging force.

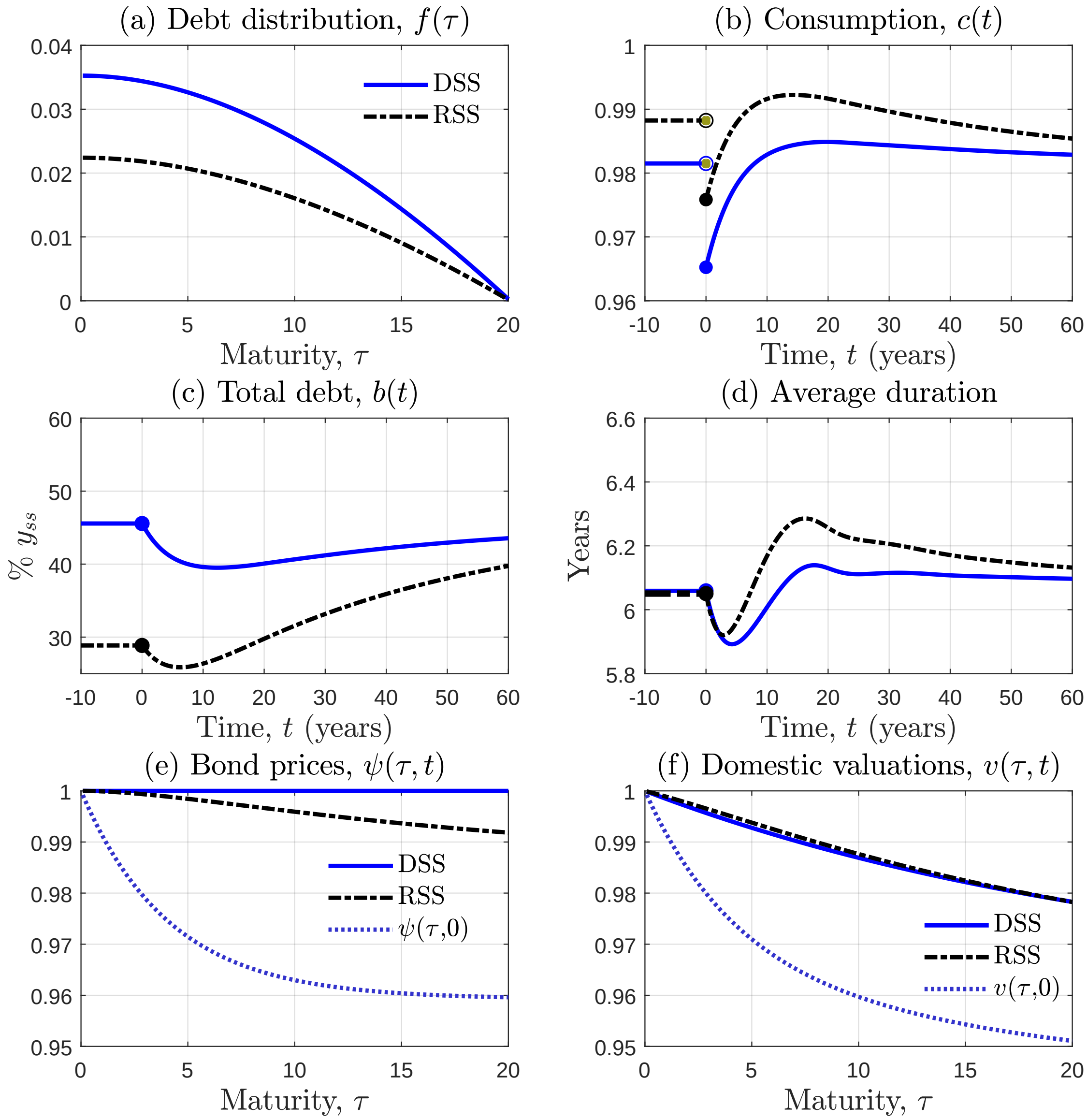

Figure 3.2: Response to a shock to interest rates. RSS is the model starting at the risky steady state and DSS at the deterministic steady state.

Once we know that valuations do not change between the RSS and the DSS and that hedging is ineffective, we conclude that the reduction in debt follows mainly from bond-price reaction, namely from the reduction in bond prices in the RSS relative to the DSS (panel e). In the RSS a future rate increase has been priced into the bonds, which has a greater effect on long-term bonds. As a result, the government avoids a high interest-rate expense and holds less debt, especially at longer maturities, which explains the slight reduction in duration (panel d).

4 Default

This section extends the model with risk to allow for the possibility of strategic default. The nature of the problem changes because now bond prices depend on government actions.

4.1 The option to default

Consider the model in section 3.1, but now, upon the realization of the shock to $X(t)$, assume that the government has the option to default. If the government exercises this option, it defaults on all debts and is barred from international markets. The value in autarky is

$$V^D(X(t^o)) \equiv \int_{t^o}^{\infty} e^{-\rho(s-t^o)} U((1-\kappa)y(s))\, ds + \varepsilon. \tag{4.1}$$

The value is the discounted utility of consuming the endowment, assuming a permanent output loss of κ, plus a a zero-mean random variable, ε. The variable ε captures randomness around the decision to default. Thus, the post-default value is also a random variable, centered around the expected value of moving into perpetual autarky with an output loss κ. Denote by $\Theta(\cdot)$ and $\theta(\cdot)$ the cumulative distribution and the probability density function of $V^D(X(t))$, respectively.[23]

If the government chooses not to default after the shock, the economy becomes the perfect-foresight economy studied in section 2. The government strategically defaults when

$$V^D(X(t^o)) > V\left[\hat{f}\left(\cdot, t^0\right), X\left(t^0\right)\right],$$

that is, if the value after default is greater than the value obtained by continuing to service its debts, which coincides with the perfect-foresight value of an initial debt profile $\hat{f}(\cdot, t^o)$. Therefore, at any given time, the default probability is

$$\mathbb{P}\left\{V^D(X(t^o)) > V\left[\hat{f}\left(\cdot, t^0\right), X\left(t^0\right)\right]\right\} = 1 - \Theta\left(V\left[\hat{f}(\cdot, t), X(t)\right]\right)$$

and the probability of repayment is $\Theta\left(V\left[\hat{f}(\cdot, t), X(t)\right]\right)$.

Market prices must be adjusted for credit risk. The corresponding pricing equation is:

$$\hat{r}^*(t)\,\hat{\psi}(\tau, t) = \delta + \frac{\partial\hat{\psi}}{\partial t} - \frac{\partial\hat{\psi}}{\partial\tau} + \phi\mathbb{E}_t^X\left[\Theta\left(V\left[\hat{f}(\cdot, t), X(t)\right]\right)\psi(\tau, t, X(t)) - \hat{\psi}(\tau, t)\right], \text{ if } t < t^o. \tag{4.2}$$

After the shock, the pricing equation is again (2.4). Prior to the shock, the pricing equation is similar to the version with risk, but now the post-shock price is multiplied by the probability of repayment. The idea is that the expected change in the price after the shock is the perfect-foresight price times the repayment probability, $\Theta(V)\,\psi$. The option to default implies that the government's actions affect bond prices, a new feature which alters the nature of the solution to the government's problem.

[23] Note that the government can hold foreign bonds ($f(\tau, t) < 0$). Default implies that these bonds are expropriated by foreign investors.

With a default option, the government's problem now becomes:

$$\max_{\{\iota(\tau,t),f(\tau,t),c(t)\}_{t\geq 0,\tau\in[0,T]}} \mathbb{E}_t \left[\int_0^{t^o} e^{-\rho t} U\left(\hat{c}\left(t\right)\right) dt + e^{-\rho t^o} \mathbb{E}^X_{t^o} \left[\max \left\{ V\left[f\left(\cdot, t^o\right), X\left(t^o\right)\right], V^D(t^o, X\left(t^0\right)) \right\} \right] \right] \tag{4.3}$$

subject to the law of motion of debt, (2.1), the budget constraint, (2.2), and the pricing equation (4.2).

In this problem, the government commits at time zero, before the realization of the shock, to a debt program. Notwithstanding, the government cannot commit to repay its debt once the shock arrives and it may default strategically. We focus on this case for three reasons. First, the problem with commitment to debt policies is a natural extension of the problems analyzed in sections 2 and 3. Second, this is a relevant case from a practical point of view. For example, it is a natural benchmark when an independent agency designs the debt policy of a sovereign country or when the government commits to a fiscal rule, which are reasonable approximations for developed economies. Third, to the best of our knowledge, the solution of the game between the government and the investors that is usually studied by the literature following Eaton and Gersovitz (1981) must rely on numerical approximations, which is beyond the scope of the current paper. This is one of the first studies to deal with the case of full commitment.[24]

4.2 Default-adjusted valuations

We adapt the framework to allow for default. The main novelty is that the government takes into account how its decisions affect market prices and the future decisions to default. To ease notation, we define $V(t) \equiv V\left[f\left(\cdot, t\right), X\left(t\right)\right]$.

Proposition 3. *(Necessary conditions) If a solution $\{c(t), \iota\left(\tau, t\right), f\left(\tau, t\right), \psi(\tau, t)\}_{t\geq 0}$ to (4.3) exists, it satisfies the same conditions of the perfect-foresight solution (Proposition 1) with valuations prior to the shock now satisfying the following PDE,*

$$\begin{aligned} \hat{r}\left(t\right)\hat{v}\left(\tau,t\right) &= \delta + \frac{\partial \hat{v}}{\partial t} - \frac{\partial \hat{v}}{\partial \tau}\dots \\ &\quad + \phi\left(\mathbb{E}^X_s\left[\left(\Theta\left(V(t)\right) + \Omega(t)\right)\frac{U'\left(c\left(t\right)\right)}{U'\left(\hat{c}\left(t\right)\right)} v\left(\tau,t\right)\right] - \hat{v}\left(\tau,t\right)\right), \\ \hat{v}\left(0,t\right) &= 1. \end{aligned} \tag{4.4}$$

[24] A recent exception is Hatchondo et al. (2018), which compares debt policy under full commitment in a model as in Eaton and Gersovitz (1981) with long-term debt, to the debt policy of the Markov Equilibrium with bonds and income as state variables.

where

$$\Omega(t) = \theta\left(V(t)\right) U'\left(\hat{c}\left(t\right)\right) \dots$$
$$\int_0^T \int_{\max\{t+m-T,0\}}^t e^{-\int_z^t (r^*(u)-\hat{r}(u))du} \psi\left(m,t\right) \frac{1}{2\bar{\lambda}} \left[1 - \left(\frac{\hat{v}\left(m+t-z,z\right)}{\hat{\psi}\left(m+t-z,z\right)}\right)^2\right] dzdm.$$

Proposition 3 shows that the option to default alters valuations, but that the principle of a simple issuance rule still holds. We can compare the expression for valuations with default, (4.4), to the expression for market prices (4.2). As discussed in section 3, without the option to default, prices and valuations would only differ in their discount rates ($\hat{r}\left(t\right)$ versus $\hat{r}^*$) and in the presence of an adjustment for risk ($\frac{U'(c(t))}{U'(\hat{c}(t))}$) in the latter. Default introduces two new features. The first is that default allows for a new form of insurance. This is captured by the repayment probability, $\Theta\left(V(t)\right)$. After-shock valuations are multiplied by the repayment probability in equation (4.4) since debt is worthless after a default. The same repayment probability appears in the market price (4.2). This captures the improvement in *insurance* provided by the option to default. As we explained in the previous section, the increase in marginal utility is driven by a self-insurance motive. The option to default reduces this motive because the probability of repayment is low when a negative shock triggers an increase in the marginal utility ratio—to put it simply, Θ offsets the effects of $\frac{U'(c(t))}{U'(\hat{c}(t))}$ in the valuations in particularly adverse states. This feature captures how default improves insurance under incomplete markets (an idea found for example in Zame, 1993).

The second feature reflects the role of *incentives*. Once the default option appears, the government cannot commit to repay. The inability to honor debts in the future impacts bonds prices. The issuance of a (τ, t)-bond increases the probability of default, which not only affects the price of that particular bond, but also the price of all the bonds that coexist with it, independently of when they were issued. This implies that the issuance of a bond at a certain date reduces revenues at different dates through a decline in bond prices, an effect which should be taken into account by the government when deciding whether to issue. The marginal impact of this reduction in revenues from each bond issuance, which we dubbed as the *revenue-echo effect*, is captured by the term Ω in equation (4.4).

We analyze the terms inside Ω to uncover an interpretation. We develop the explanation with the aid of figure 4.1. The revenue-echo $\Omega\left(t\right)$ is the product of the marginal probability of default, $\theta\left(V(t)\right) U'\left(\hat{c}\left(t\right)\right)$, and an integral described below. To understand why, consider a small issuance of a (τ_0, t_0)-bond, located at the gray dot in figure 4.1. The bond matures as time progresses, as depicted by the gray ray in the direction $(1, -1)$ that starts at the issuance point. By time t, the bond, now with a maturity $\tau = \tau_0 - (t - t_0)$, has a marginal impact on the repayment probability. If we multiply the term $\theta\left(V(t)\right) U'\left(\hat{c}\left(t\right)\right)$, inside Ω, by the term $\frac{U'(c(t))}{U'(\hat{c}(t))} v$ in the valuation, we obtain the marginal effect on the repayment probability:

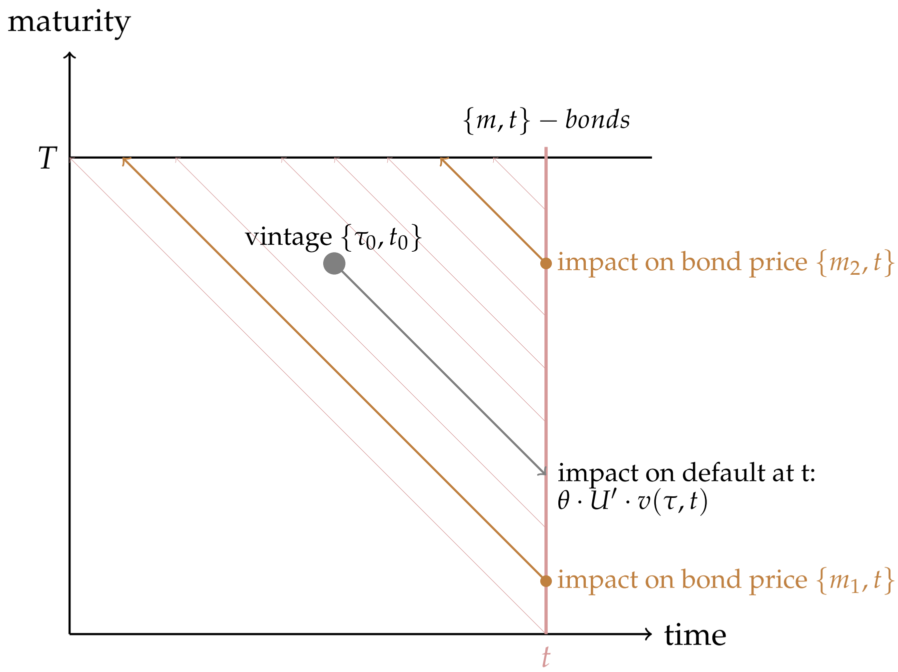

Figure 4.1: Illustration of the revenue-echo effect (axes inverted).

$$\underbrace{\theta\left(V(t)\right)U'\left(c\left(t\right)\right)}_{\text{effect on repayment probability}} \cdot v\left(\tau, t\right).$$

Hence, the term $\theta\left(V(t)\right)U'\left(c\left(t\right)\right)$ reflects the marginal effect on the repayment probability at time t.

The double integral captures how a lower repayment probability at time t impacts the revenues generated by all issuances in all moments prior to t. Notice how the range of integration covers all maturities $m \in [0, T]$ in the outer integral. The inner integral covers the relevant times prior to time t, $z \in \max\{t+m-T, 0\}$, when the marginal effect on repayment of the (τ, t)-bond affects the repayment probability of other bonds. These bonds are those that are still outstanding at time t, with a maturity m—they are depicted in the vertical line at time t. The effect of default at time t impacts past prices in an "echo effect": Any bond price at a date prior to t for a bond that is still outstanding at time t should include the discounted value of its price at t, $e^{-\int_z^t (\hat{r}^*(u))du}\psi\left(m, t\right)$ for a specific maturity m. For example, the price of a bond with maturity m, $\psi\left(m, t\right)$, affects the price of all bonds $(m+t-z, z)$ indexed by $z \in \max\{t+m-T, 0\}$. Each ray that extends from the vertical line at t depicts one such family of bonds. Thus, if we multiply the change in the repayment probability at t by $e^{-\int_z^t (\hat{r}^*(u))du}\psi\left(m, t\right)$, we obtain the reduction in the price of the $(m+t-z, z)$-bond. We can do the same for all bonds in $m \in [0, T]$, the outer integral, and past times z, the inner integral, to obtain the marginal effect that the (τ, t)-bond has on all past bond prices.

This marginal impact on past prices affects past revenues. Thus, fix a maturity m and a date t. If we want to get the effect on revenues of a change in past prices, we must multiply the change in price by $\hat{\iota}(1-(\bar{\lambda}/2)\,\hat{\iota})$, the issuance amount net of the liquidity costs. If we use the optimal issuance rule (2.12), revenues are proportional to:

$$\frac{1}{2\bar{\lambda}}\left[1-\left(\frac{\hat{v}\,(m+t-z,z)}{\hat{\psi}\,(m+t-z,z)}\right)^2\right].$$

Thus, when this term is inside the double integral, it captures the impact on past revenues of an increase in default probabilities. We bring past reductions in revenues into the current period t by multiplying by $e^{\int_z^t \hat{r}(u)du}$. The echo effect is present at any instant prior to the maturity of the bond, which is why it appears as a flow in equation (4.4). When the government considers the marginal issuance of the (τ_0, t_0)-bond, its valuation is the present value of all the echo effects Ω that last throughout the life of the bond. This reduction in revenues is part of the issuance consideration.

In Appendix G.2 we analyze the option to default without liquidity costs. In contrast to the case only with risk, the possibility of default makes it impossible to perfectly hedge. As long as the cardinality of shocks is discrete, the maturity profile is indeterminate. One extreme case of indeterminacy is that of a shock which does not produce a jump in income nor interests, but only grants a default option. Aguiar et al. (Forthcoming) studies that shock in a discrete-time model similar to ours but without commitment.

As in section 3.2, the computation of the solution in proposition 3 involves solving for a fixed point in a family of time-varying debt distributions. In this case the problem is even more complex, due to the interplay between debt management and bond prices that was absent in previous sections. However, as with the case with risk, we can obtain a solution in the RSS.

4.3 The impact of default on maturity choice

We now continue with the quantitative analysis, but incorporate default. We calibrate the output loss, κ, to 1.5 percent. This is a lower value than the 2 percent used in, for example, Aguiar and Gopinath (2006). We calibrate a lower output loss because the literature usually assumes reinsertion to capital markets after some periods. Here autarky is absorbing. A lower κ is meant to produce a similar autarky value as if we allowed for reinsertion. The distribution of ε is a logistic with coefficient ς, as is common in discrete-choice models. We set ς to 100. This number produces a default in Spain in 32 percent of the events when it is hit by an extreme shock. Coupled with the intensity of the extreme event, ϕ, the unconditional default probability is 0.6 percent per year, roughly a default every 157 years—according to Reinhart and Rogoff (2009) Spain experienced one default during the years 1877-1982. Further details on the calibration can be found in Appendix D.

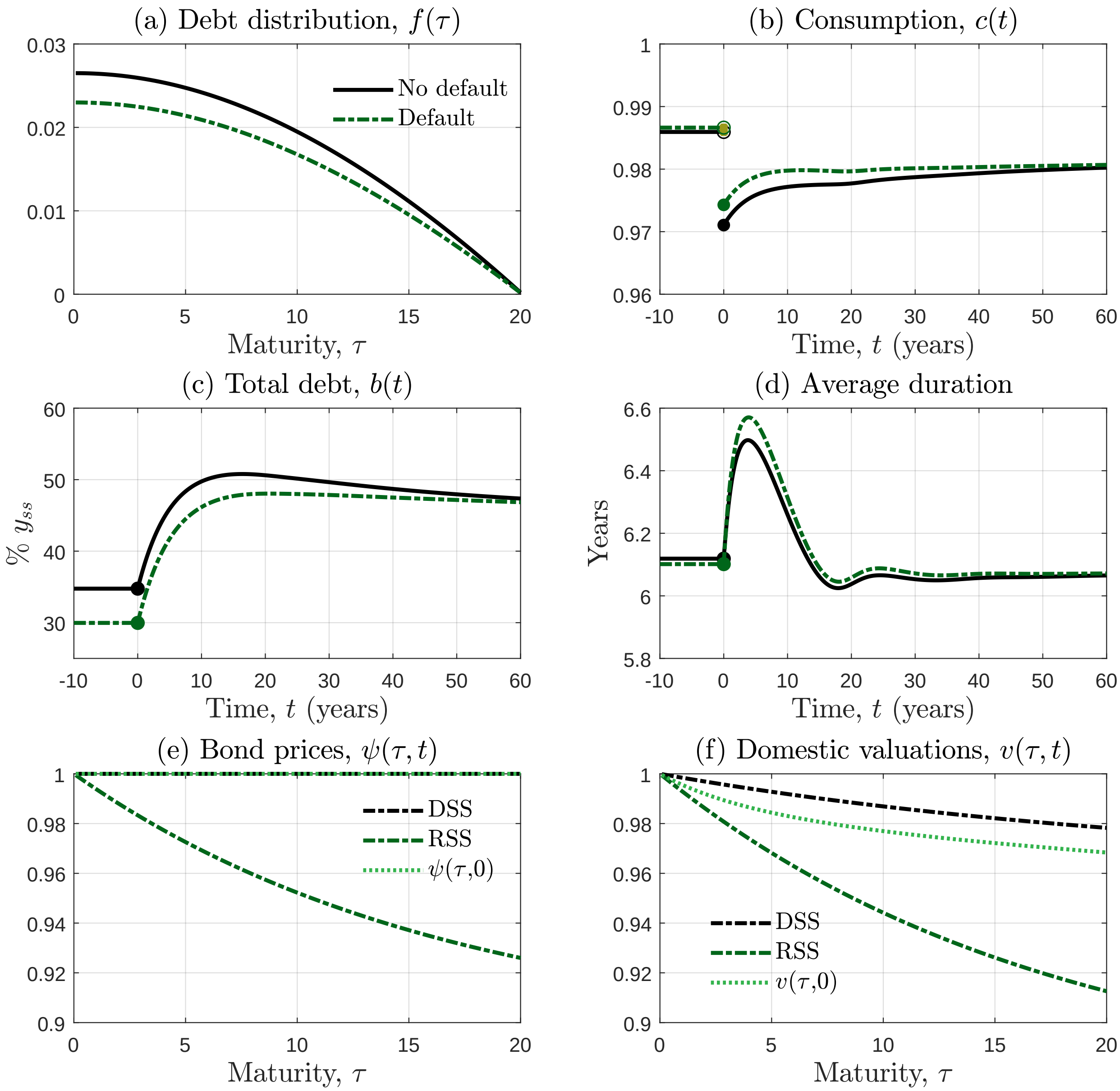

Figure 4.2: Response to a shock to income when the option to default is available. Panels (e) and (f) refer to the case with the option to default.

To gather a sense of the quantitative impact of default, figure 4.2 analyzes the RSS and the post-shock transitions in an economy with the option to default that is facing an income shock. The figure also reports the case with risk but without default, which is studied in section 3.3. We use the labels "Default," and "No default” correspondingly. Panel (e) and panel (f) show prices and valuations for the version with default at the RSS, the DSS, and the impact of the shock, $v(\tau, 0)$.

The possibility of default leads to a reduction in the level and maturity of debt compared to the case only with risk. In panel (c), we observe that the option to default produces a reduction in total debt in the RSS, from 35 percent to 30 percent. Four main forces are at play. First, as in the case only with aggregate risk, there is self-insurance. Second, despite the absence of changes to the market interest rate, bond-price reaction is operative here. This is due

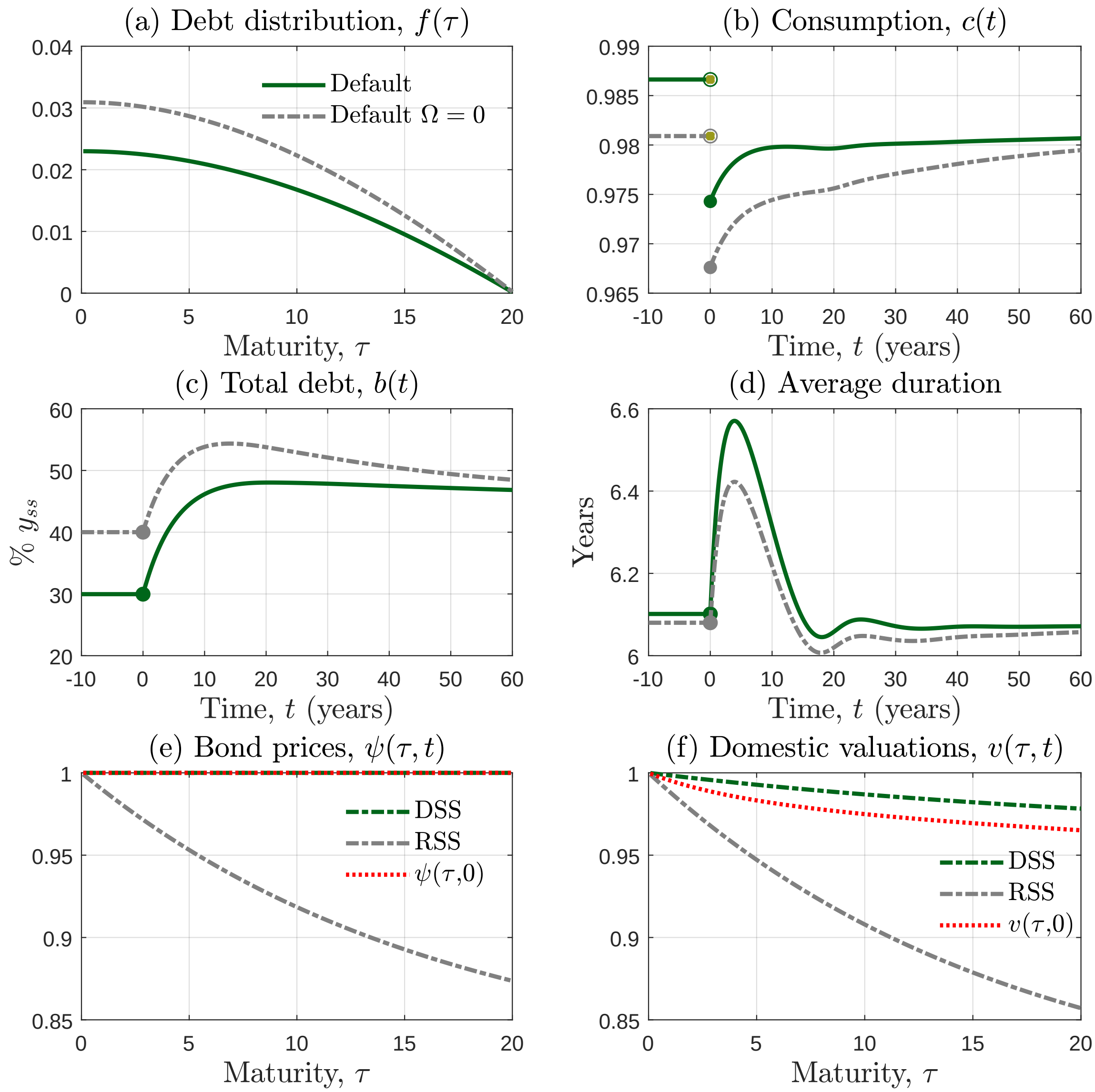

Figure 4.3: Response to a shock to income with $\Omega = 0$ when the option to default is available. Panels (e) and (f) refer to the case with $\Omega = 0$.

to the presence of the default premium, $\Theta\left(V\left[\hat{f}\left(\cdot,t\right),X\left(t\right)\right]\right)$, which reduces bond prices in the RSS (see equation 4.2). Third, as discussed above, default provides additional insurance because it limits payments in high-marginal utility paths. Fourth, the revenue-echo effect increases domestic valuations as the government internalizes how extra debt affects revenues. Self-insurance, bond-price reaction and the echo effect are all forces toward less debt, whereas the insurance provided by the default probability operates in the opposite sense.

The reduction in the debt stock is dominated by the echo effect. We can see this by calculating the analogue of figure 4.2, and comparing the solution with default with a myopic solution without echo effect ($\Omega = 0$). This comparison is reported in figure 4.3. Once we shut off the echo effect, the RSS debt stock increases from 30 to 40 percent, a level even higher than without the default option. This pattern reflects the trade-off between insurance and incentives. If we shut down the echo effect, the possibility of default improves insurance. This extra insurance is strong enough to reverse the effect of both self-insurance and bond-price reaction.

The shortening of maturities can also be read off from figure 4.2. We discussed above how, despite a constant risk-free rate, the ex-ante increase in the default premium activates the bond-price reaction force, tilting the distribution toward shorter maturities. The increase in insurance partially offsets this effect because long-term debt has higher valuations in the RSS. Back in figure 4.3 we see that, without the echo effect, maturity falls even further. At first, this result seems surprising because the echo effect penalizes longer maturity bonds. However, the echo effect reduces the debt outstanding, which lowers the default probability. Hence, the echo effect reduces the risk premium, which mitigates the bond-price reaction force that pushes the distribution toward shorter maturities.

Figure H.4 in Appendix H, compares the baseline solution with default and risk aversion with the solution assuming risk neutrality. Both solutions approximately coincide in the RSS: the debt level, consumption, and the risk premium are almost identical. This occurs because the insurance and incentives channels offset each other. This result is intuitive because the echo effect is linked to the elasticity of inter-temporal substitution through the comparison of revenues across time.

5 Extensions

To underscore the flexibility of our framework, in this section we illustrate further extensions that bring the model closer to reality or that can help us understand the cost and benefits of the introduction of alternative debt instruments, which connects this framework with topics that are currently discussed in the literature.

5.1 Alternative specifications for the liquidity costs

As a first extension, we could introduce liquidity costs that also depend on the overall stock of debt rather than only on the issued amount, in the spirit to Vayanos and Vila (2009). In that case, valuations in the deterministic version of the model should be adjusted by a term with a flavor similar to the revenue-echo-effect term discussed in section 4. The reasoning is similar, even if there is no default: Given that the issuance of a bond will affect future liquidity costs, and hence future revenues, the government should take that into account when deciding how much debt to issue at each maturity. We can also allow for a more general function of $\bar{\lambda}$, for instance one that changes with the sign of issuances or with maturity. This extension can be used to understand the impact on fiscal policy of central bank quantitative easing programs or the role of safe assets, as in (Bianchi et al., Forthcoming). Furthermore, we can assume that $\bar{\lambda}$ is a stochastic process, which could be interpreted as rollover risk.

5.2 Finite issuances

Through sections 2, 3 and 4, the sovereign actively issues in a continuum of maturities. In practice, however, countries issue only at a discrete number of maturities. One example are the issuances of Spain, as we discuss in Appendix E. Here we show how the model is adaptable to allow for issuances at a discrete set of points. We set the liquidity coefficient $\bar{\lambda}(N)$ to be finite only for a discrete number of maturities $\{\tau_1, \tau_2, ..., \tau_N\}$ as well as a function of the total number of available maturities, N.[25]

For the deterministic, risk, and default cases, simple extensions to Propositions 1, 2 and 3 imply that the optimal issuance rule would be:

$$\iota(\tau, t) = \begin{cases} \frac{1}{\bar{\lambda}(N)} \cdot \frac{\psi(\tau,t) - v(\tau,t)}{\psi(\tau,t)} & \tau \in \{\tau_1, \tau_2, ..., \tau_N\} \\ 0 & \tau \notin \{\tau_1, \tau_2, ..., \tau_N\}, \end{cases}$$

without changes to the valuation formulas for the three cases. To render reasonable comparisons, the liquidity coefficient, $\bar{\lambda}(N)$, has to be adapted to the total number of maturities, N. This can be done by keeping the overall customer flow constant and spreading it equally across maturities. We can use this formulation to study, for instance, the welfare gains of increasing the set of maturities at which the government issues.

In Appendix E, we describe how we also observe the pattern of increasing issuances by maturity, but that the pattern holds for two groups of bonds—below and above one year. It is easy to adapt the model to reproduce this profile, for example, by introducing a liquidity coefficient that varies within groups of bonds. This extension can capture segmented markets or institutional features that affect the order flow at different maturities.

5.3 Consols

To keep the state space manageable, most quantitative applications on debt management employ bonds that mature probabilistically, also known as *consols* (see, for example, Leland and Toft, 1996, or Hatchondo and Martinez, 2009). Perpetuities with different coupons, proposed by Cochrane (2015) and studied from a normative point of view by Debortoli et al. (2018), have also received some attention. We can adapt the environment to compare the steady state and transitions. We use the "c" superscript to refer to the consols. For concreteness, we explain the case with perfect-foresight consols, but noting substantial would change if we were to add risk.

[25] Technically, issuances have to be modeled with Dirac delta functions at the issuance points, obtained as a limit where we shrink intervals. In terms of computations, this technical detail does not matter because the maturity space is discretized.

The market price at time $t$of a consol maturing at a rate $\frac{1}{\tilde{\tau}}$, $\psi^{c}\left(\tilde{\tau},t\right)$, satisfies:

$$r^{*}(t)\psi^{c}\left(\tilde{\tau},t\right)=(\frac{1}{\tilde{\tau}}+\delta)-\frac{1}{\tilde{\tau}}\psi^{c}\left(\tilde{\tau},t\right)+\frac{\partial\psi^{c}(\tilde{\tau},t)}{\partial t}.$$

with $\lim_{\tilde{\tau}\to 0}\psi^{c}(\tilde{\tau},t)=1$. The first term in in this valuation equation, $\left(\frac{1}{\tilde{\tau}}+\delta\right)$, denotes the coupons δ and a principal repayment $\frac{1}{\tilde{\tau}}$. The second term captures the maturity rate of the bond, and the third term captures the change in bond prices over time. The government's budget constraint and law of motion of debt are:

$$c\left(t\right)=y\left(t\right)+\int_{0}^{T}\left[q^{c}(\tilde{\tau},t,\iota)\iota^{c}(\tilde{\tau},t)-\left(\frac{1}{\tilde{\tau}}+\delta\right)f^{c}\left(\tilde{\tau},t\right)\right]d\tilde{\tau},$$
$$\frac{\partial f^{c}(\tilde{\tau},t)}{\partial t}=\iota^{c}(\tilde{\tau},t)-\frac{1}{\tilde{\tau}}f^{c}(\tilde{\tau},t).$$

The solution to this problem entails the same issuance rule as in the case of finite-life bonds, equation (2.12), but domestic valuations are now given by

$$r\left(t\right)v^{c}\left(\tilde{\tau},t\right)\;=\;\left(\frac{1}{\tilde{\tau}}+\delta\right)-\frac{1}{\tilde{\tau}}v^{c}\left(\tilde{\tau},t\right)+\frac{\partial v^{c}}{\partial t},\ \frac{1}{\tilde{\tau}}\in(0,T],$$

with $\lim_{\frac{1}{\tilde{\tau}}\to 0}v(\tilde{\tau},t)=1$. At a steady state with positive consumption, assuming that $\delta=r^{*}_{ss}$, we obtain that prices and valuations should satisfy $\psi^{c}_{ss}(\tilde{\tau})=1$ and $v^{c}_{ss}(\tilde{\tau})=\left(r^{*}_{ss}+\frac{1}{\tilde{\tau}}\right)/\left(\rho+\frac{1}{\tilde{\tau}}\right)$. The steady-state debt profile of consol coefficients is:

$$f^{c}_{ss}\left(\tilde{\tau}\right)=\frac{\tilde{\tau}\left(\rho-r^{*}_{ss}\right)}{\bar{\lambda}\left(\rho+\frac{1}{\tilde{\tau}}\right)},$$

and the associated profile of future debt payments, i.e., the equivalent object to the debt profile in the case of finite-life bonds, is

$$f^{c}_{ss}\left(\tau\right)=\int_{0}^{T}\left(\frac{1}{\tilde{\tau}}+\delta\right)e^{-\tau/\tilde{\tau}}f^{c}_{ss}\left(\tilde{\tau}\right)d\tilde{\tau},$$

which has a different shape than that under finite-life bonds. The conclusion is that, in the presence of liquidity costs, the choice between consols and standard debt has important empirical implications.

6 Conclusions

The aim of this paper is to present a framework to study debt management problems with debt instruments that closely resemble the ones that governments issue in reality. The main challenge of these problems is that the state variable is a distribution. A central feature of the framework is the presence of liquidity costs that limit the possibility of immediate rebalancing across maturities. The paper showcases a general principle: optimal issuances are proportional to the gap between the market price of debt and its domestic valuation. In the presence of risk and default, prices and valuations are adapted, but the principle remains the same. All in all, the paper highlights classic forces and uncovers new ones that shape the optimal debt-maturity distribution.

As a first step in a less explored direction, the framework naturally faces limitations. We are optimistic about the prospect for improvement. One limitation is that we can obtain exact solutions when shocks occur only once. This captures, up to a first order, expected impulse responses. An extension that admits recurrent shocks faces the same computational challenges as heterogeneous agent models with aggregate shocks. Second, we consider a specific form of liquidity costs. However, as we stress throughout the paper, the framework can be extended to allow for more general liquidity costs that, for instance, vary with maturity and the amount of outstanding debt. Finally, when we study default, we assume that the government can commit to an issuance path. The challenge of a solution without commitment is that we will have to compute terms that capture how current actions disciplines the future self. We leave these interesting issues for future research.

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Appendix: A Framework for Debt-Maturity Management

Contents

A Equivalence between PDE and integral formulations

Valuations and prices are a given by continuous-time net present value formulae. Their PDE representation is the analogue of the recursive representation in discrete time and the integral formulation is the equivalent of the sequence summations. The solutions to each PDE can be recovered easily via the method of characteristics or as an immediate application of the Feynman-Kac formula. All of the PDEs in this paper have an exact solution contained in table A.

Price (PF)	PDE	$r^*(t)\,\psi(\tau,t) = \delta + \frac{\partial\psi}{\partial t} - \frac{\partial\psi}{\partial\tau};\ \psi(0,t) = 1$
	Integral	$e^{-\int_t^{t+\tau} r^*(u)du} + \delta\int_t^{t+\tau} e^{-\int_t^s r^*(u)du}ds$
Valuation (PF)	PDE	$r(t)\,v(\tau,t) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial\tau};\ v(0,t) = 1$
	Integral	$e^{-\int_t^{t+\tau} r(u)du} + \delta\int_t^{t+\tau} e^{-\int_t^s r(u)du}ds$
Price (risk)	PDE	$\hat{r}^*(t)\,\hat{\psi}(\tau,t) = \delta + \frac{\partial\hat{\psi}}{\partial t} - \frac{\partial\hat{\psi}}{\partial\tau} + \phi\mathbb{E}_t^X\left[\psi(\tau,t,X_t) - \hat{\psi}(\tau,t)\right];\ \hat{\psi}(0,t) = 1$
	Integral	$e^{-\int_t^{t+\tau}(\hat{r}^*(u)+\phi)du} + \int_t^{t+\tau}\left(\delta + \hat{\psi}(\tau-s,t+s)\right)e^{-\int_t^s(\hat{r}^*(u)+\phi)du}ds$
Valuation (risk)	PDE	$\hat{r}^*(t)\,\hat{v}(\tau,t) = \delta + \frac{\partial\hat{v}}{\partial t} - \frac{\partial\hat{v}}{\partial\tau} + \phi\mathbb{E}_t^X\left[v(\tau,t,X_t) - \hat{v}(\tau,t)\right];\ \hat{v}(0,t) = 1$
	Integral	$e^{-\int_t^{t+\tau}(\hat{r}(u)+\phi)du} + \int_t^{t+\tau}\left(\delta + v(\tau-s,t+s)\,\frac{U'(c(t+s))}{U'(\hat{c}(t+s))}\right)e^{-\int_t^s(\hat{r}(u)+\phi)du}ds$
Price (default)	PDE	$\hat{r}^*(t)\,\hat{\psi}(\tau,t) = \delta + \frac{\partial\hat{\psi}}{\partial t} - \frac{\partial\hat{\psi}}{\partial\tau} + \phi\mathbb{E}_t^X\left[\Phi\left(V\left[\hat{f}(\cdot,t)\right]\right)\psi(\tau,t,X_t) - \hat{\psi}(\tau,t)\right];\ v(0,t) = 1$
	Integral	$e^{-\int_t^{t+\tau}(\hat{r}^*(u)+\phi)du} + \int_t^{t+\tau} e^{-\int_t^s(\hat{r}^*(u)+\phi)du}\left(\delta + \phi\mathbb{E}_s^X\left[\Theta\left(V\left[\hat{f}(\cdot,t)\right]\right)\psi(\tau-s,t+s)\right]\right)ds$
Valuation (default)	PDE	$\hat{r}^*(t)\,\hat{v}(\tau,t) = \delta + \frac{\partial\hat{v}}{\partial t} - \frac{\partial\hat{v}}{\partial\tau} + \phi\left(\mathbb{E}_s^X\left[(\Theta(V(t)) + \Omega(t))\,\frac{U'(c(t))}{U'(\hat{c}(t))}v(\tau,t)\right] - \hat{v}(\tau,t)\right);\ \hat{v}(0,t) = 1$
	Integral	$e^{-\int_t^{t+\tau}(\hat{r}^*(s)+\phi)ds} + \int_t^{t+\tau} e^{-\int_t^s(\hat{r}^*(u)+\phi)du}\left(\delta + \phi\mathbb{E}_s^X\left[\left(\Omega(t+s) + \Theta\left(V\left[\hat{f}(\cdot,t+s)\right]\right)\right)\frac{U'(c(t+s))}{U'(\hat{c}(t+s))}v(\tau-s,t+s)\right]\right)ds$
Debt Profile	PDE	$\frac{\partial f}{\partial t} = \iota(\tau,t) + \frac{\partial f}{\partial\tau}$
	Integral	$f(\tau,t) = \int_\tau^{min\{T,\tau+t\}}\iota(s,t+\tau-s)ds + \mathbb{I}[T > t+\tau]\cdot f(0,\tau+t)$

Table 1: Equivalence between PDE and integral formulations.

B Micro model of liquidity costs

B.1 Environment

Here we describe the microeconomic model of the liquidity costs in more detail. This is a wholesale retail model of the secondary market of sovereign bonds. Without loss of generality, we focus on the issuance of $\iota(\tau, t)$-bonds at time t with maturity τ. We define s as the amount of time passed since the auction. The outstanding amount of bonds in the hands of an atomistic banker, after a period of time s has passed after the issuance of the bond is:

$$I\left(s; \iota(\tau, t)\right) = \max(\iota(\tau, t) - \mu y_{ss} \cdot s, 0).$$

This implies that the bond inventory is exhausted by time:

$$\bar{s} = \frac{\iota(\tau, t)}{\mu y_{ss}}.$$

We consider that individual orders arrive randomly according to a Poisson distribution. The intensity at which bonds are sold per unit of time is given by:

$$\begin{aligned}
\gamma^{(t,\tau)}\left(s\right) &= \frac{\mu y_{ss}}{I\left(s; \iota(\tau, t)\right)} \\
&= \frac{1}{\bar{s} - s} \text{ for } s \in \left[0, \min\{\tau, \bar{s}\}\right).
\end{aligned}$$

This intensity $\gamma^{(t,\tau)}\left(s\right)$ is defined only between $[0, \min\{\tau, \bar{s}\})$, because after the bond matures or after the stock is exhausted, there is no further selling.

B.2 Valuations

Investor's valuation. At time $t + s$, after a period of time s has passed since the auction, the time to maturity is $\tau' = \tau - s$. The valuation of the bond by investors is defined as:

$$\psi^{(t,\tau)}(\tau', s) \equiv \psi(\tau - s, t + s).$$

They are risk neutral and discount future payoffs at the international market rate. Hence, the price equation satisfies the PDE (2.4):

$$r^*(t + s)\psi^{(t,\tau)}(\tau', s) = \delta - \frac{\partial \psi^{(t,\tau)}}{\partial \tau'} + \frac{\partial \psi^{(t,\tau)}}{\partial t},$$

with the terminal condition of $\psi^{(t,\tau)}(0, s) = 1$.

Banker's valuation. Now consider the valuation of the cash flows of the bond from the perspective of the banker $q^{(t,\tau)}(\tau', s)$. Bankers are risk neutral but have a higher cost of capital. At each moment $t + s$ bankers meet investors and sell at a price $\psi^{(t,\tau)}(\tau', s)$. The valuation of the bankers, $q^{(t,\tau)}(\tau', s)$ satisfies:

$$(r^*(t + s) + \eta)q^{(t,\tau)}(\tau', s) = \delta - \frac{\partial q^{(t,\tau)}}{\partial \tau'} + \frac{\partial q^{(t,\tau)}}{\partial t} + \gamma^{(t,\tau)}\left(s\right)\left(\psi^{(t,\tau)}(\tau', s) - q^{(t,\tau)}(\tau', s)\right). \tag{B.1}$$

This expression takes this form because the banker extracts surplus $\left(\psi^{(t,\tau)}(\tau', s) - q^{(t,\tau)}(\tau', s)\right)$ when he is matched to an investor. Before a match, bankers earn the flow utility, but upon a match, their value jumps to $\psi^{(t,\tau)} - q^{(t,\tau)}$. This jump arrives with endogenous intensity $\gamma^{(t,\tau)}\left(s\right)$. The complication with this PDE is its terminal condition.

If $\bar{s} \leq \tau$, the PDE's terminal condition is given by $q^{(t,\tau)}(\tau', \bar{s}) = \psi^{(t,\tau)}(\tau', \bar{s})$. If $\bar{s} > \tau$, the corresponding terminal condition is $q^{(t,\tau)}(0,s) = 1$, since by the expiration date, the investor is paid the principal equal to 1.

Competitive Auction Price. On the date of the auction, $s = 0$, $\tau' = \tau$, the banker pays his expected bond valuation, hence the bond price demand faced by the government is:

$$q(\iota, \tau, t) \equiv q^{(t,\tau)}(\tau, 0).$$

This is because banks have free entry into the auction. In equation (2.3) we expressed $q(\iota, \tau, t)$ as $q(\iota, \tau, t) = \psi(\tau, t) - \lambda(\iota, \tau, t)$. Thus,

$$\lambda(\iota, \tau, t) = \psi(\tau, t) - q(\iota, \tau, t),$$

is the object we are trying to find.

B.3 Solution

We now provide a first order linear approximation for the price at the auction, $q(\iota, \tau, t)$, for small issuances. The result is given by the following proposition:

Proposition 4. *A first-order Taylor expansion around $\iota = 0$ yields a linear auction price:*

$$q(\iota, \tau, t) \approx \psi(\tau, t) - \frac{1}{2} \frac{\eta}{\mu y_{ss}} \psi(\tau, t) \iota(\tau, t). \tag{B.2}$$

Thus, the approximate liquidity cost function is $\lambda(\tau, t, \iota) \approx \frac{1}{2} \bar{\lambda} \psi(\tau, t) \iota(\tau, t)$ where the price impact is given by $\bar{\lambda} = \frac{\eta}{\mu y_{ss}}$.

Proof. Step 1. Exact solutions. The solution to $q(\iota, \tau, t)$ falls into one of two cases. Case 1. If $\bar{s} \leq \tau$, then:

$$q(\iota, \tau, t) = \frac{\int_0^{\bar{s}} e^{-\int_0^{v} (r^*(t+u)+\eta) du} \left(\delta(\bar{s} - v) + \psi(\tau - v, t + v)\right) dv}{\bar{s}}. \tag{B.3}$$

Case 2. If $\bar{s} > \tau$, then:

$$\begin{aligned} q(\iota, \tau, t) \quad &= \quad \int_0^{\tau} e^{-\int_0^{s} (r^*(t+u)+\eta) du} \left(\frac{\delta(\bar{s} - v) + \psi(\tau - v, t + v)}{\bar{s}} \right) dv \\ &\quad + e^{-\int_0^{\tau} (r^*(t+u)+\eta) du} \frac{(\bar{s} - \tau)}{\bar{s}}. \end{aligned} \tag{B.4}$$

We solve the PDE for q depending on the corresponding terminal conditions, $q^{(t,\tau)}(\tau', \bar{s}) = \psi^{(t,\tau)}(\tau', \bar{s})$ and $q^{(t,\tau)}(0, s) = 1$.

Case 1. Consider the first case. The general solution to the PDE equation for $q^{(t,\tau)}(\tau', s)$ is,

$$\begin{aligned} &\int_0^{\bar{s}-s} e^{-\int_0^{v} (r^*(t+u)+\eta+\gamma(u)) du} \left(\delta + \gamma(s+v) \psi(\tau - v, t + v)\right) dv \quad + \\ &\qquad e^{-\int_0^{\bar{s}-s} (r^*(t+u)+\eta+\gamma(u)) du} \psi(\tau' - (\bar{s} - s), t + (\bar{s} - s)). \end{aligned} \tag{B.5}$$

This can be checked by taking partial derivatives with respect to time and maturity and applying Leibniz's rule.[26] Consider the exponentials that appear in both terms of equation (B.5). These can be decomposed into

[26]Notice that we have directly replaced the value $\psi^{(t,\tau)}(\tau', s) = \psi(\tau - s, t + s)$.

$e^{-\int_0^v [r^*(t+u)+\eta]du} e^{-\int_0^v \gamma(u)du}$. Then, by definition of γ we have:

$$\begin{aligned} e^{-\int_0^v \gamma(u)du} &= e^{-\int_0^v \frac{1}{\bar{s}-u}du} \\ &= \frac{(\bar{s}-v)}{\bar{s}}. \end{aligned} \tag{B.6}$$

Thus, using (B.6) in (B.5) we can re-express it as:

$$\begin{aligned} q^{(t,\tau)}(\tau',s) &= \int_0^{\bar{s}-s} e^{-\int_0^v (r^*(t+u)+\eta)du} \frac{(\bar{s}-v)}{\bar{s}} \left(\delta + \gamma(s+v)\psi(\tau-v,t+v)\right) dv \\ &+ e^{-\int_0^{\bar{s}-s}(r^*(t+u)+\eta)du} \frac{s}{\bar{s}} \psi(\tau' - (\bar{s}-s), t+\bar{s}). \end{aligned}$$

When we evaluate this expression at $s=0$, $\tau'=\tau$, and we replace $\gamma(v) = \frac{1}{\bar{s}-v}$, we arrive at:

$$\begin{aligned} q(\iota,\tau,t) &\equiv q^{(t,\tau)}(\tau,0) \\ &= \int_0^{\bar{s}} e^{-\int_0^v (r^*(t+u)+\eta)du} \left(\frac{(\bar{s}-v)}{\bar{s}}\delta + \frac{\psi(\tau-v,t+v)}{\bar{s}} \right) dv. \end{aligned}$$

Case 2. The proof in the second case runs parallel to Case 1 above. The general solution to the PDE in this case is:

$$\begin{aligned} q^{(t,\tau)}(\tau',s) &= \int_0^{\tau'} e^{-\int_0^v (r^*(t+u)+\eta+\gamma(u))du} \left(\delta + \gamma(s+v)\psi(\tau-v,t+v)\right) dv \\ &+ e^{-\int_0^v (r^*(t+u)+\eta+\gamma(u))du}. \end{aligned}$$

When we evaluate this expression at $s=0$, $\tau'=\tau$:

$$\begin{aligned} q(\iota,\tau,t) &= \int_0^{\tau} e^{-\int_0^s (r^*(t+u)+\eta)du} \frac{(\bar{s}-v)}{\bar{s}} \left(\delta + \frac{\psi(\tau-v,t+v)}{(\bar{s}-v)} \right) dv \\ &+ e^{-\int_0^\tau (r^*(t+u)+\eta)du} \frac{(\bar{s}-\tau)}{\bar{s}}. \end{aligned}$$

Step 2. Limit Behavior of $q(\iota,\tau,t)$. Price with zero issuances. Consider the limit $\iota(\tau,t) \longrightarrow 0$ for any $\tau > 0$, which implies that $\bar{s} \longrightarrow 0$. For both Case 1 and Case 2, equations (B.3) and (B.4),[27] it holds that:

$$\lim_{\iota(\tau,t)\longrightarrow 0} q(\iota,\tau,t) = \lim_{\bar{s}\longrightarrow 0} \frac{\int_0^{\bar{s}} e^{-\int_0^s (r^*(t+u)+\eta)du} \left(\delta(\bar{s}-s) + \psi(\tau-s,t+s)\right) ds}{\bar{s}}.$$

Now, both the numerator and the denominator converge to zero as we take the limits. Hence, by L'Hôpital's rule, the limit of the price is the limit of the ratio of derivatives. The derivative of the numerator is obtained via Leibniz's rule and thus,

$$\begin{aligned} \lim_{\iota(\tau,t)\longrightarrow 0} q(\iota,\tau,t) &= \lim_{\bar{s}\longrightarrow 0} \frac{\left[e^{-\int_0^s (r^*(t+u)+\eta)du} \left(\delta(\bar{s}-s) + \psi(\tau-s,t+s)\right)\right] |_{s=\bar{s}}}{1} \\ &= \lim_{\bar{s}\longrightarrow 0} e^{-\int_0^{\bar{s}} (r^*(t+u)+\eta)du} \psi(\tau-\bar{s},t+\bar{s}) \\ &= \psi(\tau,t). \end{aligned}$$

Step 3. Linear approximation of $q(\iota,\tau,t)$. The first order approximation of the function $q(\iota,\tau,t)$, the price at the

[27]For every $\tau < \bar{s}$, i.e. in Case 2, it will be analogous since we are taking the limit when $\bar{s}$ converges to zero.

auction, around $\iota = 0$ is given by:

$$q(\iota,\tau,t) \simeq q(\iota,\tau,t) \mid_{\iota=0} + \frac{\partial q(\iota,\tau,t)}{\partial \iota} \mid_{\iota=0} \iota(\tau,t).$$

We computed the first term in step 2. It is given by $\psi(\tau,t)$. Thus, our objective will be to obtain $\frac{\partial q(\iota,\tau,t)}{\partial \iota} \mid_{\iota=0}$. Observe that by definition of $\bar{s}$, it holds that:

$$\begin{aligned}\frac{\partial q(\iota,\tau,t)}{\partial \iota} &= \frac{\partial \bar{s}}{\partial \iota}\frac{\partial q(\iota,\tau,t)}{\partial \bar{s}} \\ &= \frac{1}{\mu y_{ss}}\frac{\partial q(\iota,\tau,t)}{\partial \bar{s}},\end{aligned}$$

where we have applied the fact that $\bar{s} = \frac{\iota(\tau,t)}{\mu y_{ss}}$. For further reference, note that

$$\frac{\partial q(\iota,\tau,t)}{\partial \iota} \mid_{\iota=0} = \lim_{\bar{s}\to 0} \frac{\partial q(\iota,\tau,t)}{\partial \bar{s}} \frac{1}{\mu y_{ss}}. \tag{B.7}$$

Step 3.1. Derivative $\frac{\partial q(\iota,\tau,t)}{\partial \bar{s}}$. Consider the price function corresponding to Case 1. The derivative of the price function with respect to $\bar{s}$ is given by:

$$\begin{aligned}\frac{\partial q(\iota,\tau,t)}{\partial \bar{s}} &= \frac{\partial}{\partial \bar{s}}\left(\frac{\int_0^{\bar{s}} e^{-\int_0^s (r^*(t+u)+\eta)du}\left(\delta(\bar{s}-s)+\psi(\tau-s,t+s)\right)ds}{\bar{s}}\right) \\ &= \frac{e^{-\int_0^{\bar{s}}(r^*(t+u)+\eta)du}\psi(\tau-\bar{s},t+\bar{s}) + \int_0^{\bar{s}} \delta e^{-\int_0^s (r(t+u)+\eta)du}ds}{\bar{s}} \\ &\quad - \frac{\int_0^{\bar{s}} e^{-\int_0^s (r^*(t+u)+\eta)du}\left(\delta(\bar{s}-s)+\psi(\tau-s,t+s)\right)ds}{\bar{s}^2} \\ &= \frac{e^{-\int_0^{\bar{s}}(r^*(t+u)+\eta)du}\psi(\tau-\bar{s},t+\bar{s}) + \int_0^{\bar{s}} \delta e^{-\int_0^s (r^*(t+u)+\eta)du}ds - q(\iota,\tau,t)}{\bar{s}}.\end{aligned} \tag{B.8}$$

Note that in the last line we used the definition of $q(\iota,\tau,t)$ as given for Case 1.

Step 3.2. Re-writing the limit of $\frac{\partial q(\iota,\tau,t)}{\partial \bar{s}}$. To obtain $\frac{\partial q(\iota,\tau,t)}{\partial \iota} \mid_{\iota=0}$ we compute $\lim_{\bar{s}\to 0} \frac{\partial q(\iota,\tau,t)}{\partial \bar{s}}$ using equation (B.8). In equation (B.8) both the numerator and denominator converge to zero as $\bar{s} \longrightarrow 0$.[28] Thus, we employ L'Hôpital's rule to obtain the derivative of interest. The derivative of the denominator is 1. Thus, the limit of (B.8) is now given by:

$$\lim_{\bar{s}\longrightarrow 0} \frac{\partial q(\iota,\tau,t)}{\partial \bar{s}} = \lim_{\bar{s}\longrightarrow 0} \frac{\partial}{\partial \bar{s}}\left[e^{-\int_0^{\bar{s}}(r^*(t+u)+\eta)du}\psi(\tau-\bar{s},t+\bar{s}) + \int_0^{\bar{s}} \delta e^{-\int_0^s (r^*(t+u)+\eta)du}ds - q(\iota,\tau,t)\right]. \tag{B.9}$$

[28]The limits of the three terms in the numerator of equation (B.8) are respectively:

$$\begin{aligned}\lim_{\bar{s}\longrightarrow 0} \int_0^{\bar{s}} e^{-\int_0^s (r(t+u)+\eta)du}ds &= 0, \\ \lim_{\bar{s}\longrightarrow 0} e^{-\int_0^{\bar{s}}(r(t+u)+\eta)du}\psi(\tau-\bar{s},t+\bar{s})) &= \psi(\tau,t), \\ \lim_{\bar{s}\longrightarrow 0} q(\iota,\tau,t) &= \psi(\tau,t).\end{aligned}$$

Step 3.3. Consider the first two terms of (B.9). Applying Leibniz's rule:

$$\lim_{\bar{s}\longrightarrow 0}\left\{\left(-\frac{\partial}{\partial\tau}\psi(\tau-\bar{s},t+\bar{s})+\frac{\partial}{\partial t}\psi(\tau-\bar{s},t+\bar{s})-(r^*(t+\bar{s})+\eta)\psi(\tau-\bar{s},t+\bar{s})\right)e^{-\int_0^{\bar{s}}(r^*(t+u)+\eta)du} \right.$$
$$\left. +\delta e^{-\int_0^{\bar{s}}(r^*(t+u)+\eta)du}\right\}.$$

The previous limit is given by:

$$-\frac{\partial}{\partial\tau}\psi(\tau,t)+\frac{\partial}{\partial t}\psi(\tau,t)-(r^*(t)+\eta)\psi(\tau,t)+\delta.$$

Using the valuation of the international investors, we can rewrite the previous equation as:

$$\begin{aligned}-\frac{\partial}{\partial\tau}\psi(\tau,t)+\frac{\partial}{\partial t}\psi(\tau,t)-(r^*(t)+\eta)\psi(\tau,t)+\delta &= r^*(t)\psi(\tau,t)-(r^*(t)+\eta)\psi(\tau,t)\\ &= -\eta\psi(\tau,t). \end{aligned} \tag{B.10}$$

This the first two terms of the limit of $\frac{\partial q(\iota,\tau,t)}{\partial\bar{s}}$ are equal to $-\eta\psi(\tau,t)$. Computing the limit of $\frac{\partial q(\iota,\tau,t)}{\partial\bar{s}}$: last term. The last term of (B.9) is given by

$$\begin{aligned}-\lim_{\bar{s}\longrightarrow 0}\frac{\partial q(\iota,\tau,t)}{\partial\bar{s}} &= -\lim_{\bar{s}\longrightarrow 0}\frac{\partial q(\iota,\tau,t)}{\partial\iota}\frac{\partial\iota}{\partial\bar{s}}\\ &= -\frac{\partial q(\iota,\tau,t)}{\partial\iota}\mid_{\iota=0}\mu y_{ss}, \end{aligned} \tag{B.11}$$

where we used (B.7). Thus, from (B.10) and (B.11), the derivative (B.8) is given by:

$$\lim_{\bar{s}\longrightarrow 0}\frac{\partial q(\iota,\tau,t)}{\partial\bar{s}} = -\frac{\partial q(\iota,\tau,t)}{\partial\iota}\mid_{\iota=0}\mu y_{ss}-\eta\psi(\tau,t). \tag{B.12}$$

Plugging (B.12) in (B.7) we obtain that:

$$\frac{\partial q(\iota,\tau,t)}{\partial\iota}\mid_{\iota=0} = \left(-\mu y_{ss}\frac{\partial q(\iota,\tau,t)}{\partial\bar{\iota}}\mid_{\iota=0}-\eta\psi(\tau,t)\right)\frac{1}{\mu y_{ss}}.$$

Rearranging terms, we conclude that:

$$\frac{\partial q(\iota,\tau,t)}{\partial\iota}\mid_{\iota=0} = -\frac{\eta\psi(\tau,t)}{2\mu y_{ss}}. \tag{B.13}$$

Step 4. Taylor expansion. A first-order Taylor expansion around zero emissions yields:

$$\begin{aligned} q(\iota,\tau,t) &\simeq q(\iota,\tau,t)\mid_{\iota=0}+\frac{\partial q(\iota,\tau,t)}{\partial\iota}\mid_{\iota=0}\iota(\tau,t),\\ &= \psi(\tau,t)-\frac{\eta\psi(\tau,t)}{2\mu y_{ss}}\iota(\tau,t). \end{aligned}$$

where we used (B.13). We can define price impact as $\bar{\lambda}=\frac{\eta}{\mu y_{ss}}$. This concludes the proof. □

C Proofs

C.1 Proof of Proposition 1

Proof. First we construct a Lagrangian on the space of functions g such that are Lebesgue integrable, $\left\| e^{-\rho t/2} g(\tau,t) \right\|^2 < \infty$. The Lagrangian, after replacing $c(t)$ from the budget constraint, is:

$$\begin{aligned} \mathcal{L}[\iota, f] &= \int_0^\infty e^{-\rho t} U\left(y(t) - f(0,t) + \int_0^T [q(\tau,t,\iota)\,\iota(\tau,t) - \delta f(\tau,t)]\, d\tau \right) dt \\ &\quad + \int_0^\infty \int_0^T e^{-\rho t} j(\tau,t) \left(-\frac{\partial f}{\partial t} + \iota(\tau,t) + \frac{\partial f}{\partial \tau} \right) d\tau dt, \end{aligned}$$

where $j(\tau,t)$ is the Lagrange multiplier associated to the law of motion of debt.

We consider a perturbation $h(\tau,t)$, $e^{-\rho t} h \in L^2([0,T] \times [0,\infty))$, around the optimal solution. Since the initial distribution f_0 is given, any feasible perturbation must satisfy $h(\tau,0) = 0$. In addition, we know that $f(T,t) = 0$ because $f(T^+,t) = 0$ (by construction) and issuances are infinitesimal. Thus, any admissible variation must also feature $h(T,t) = 0$. At an optimal solution f, the Lagrangian must satisfy$\mathcal{L}[\iota, f] \geq \mathcal{L}[\iota, f + \alpha h]$ for any perturbation $h(\tau,t)$.

Taking the derivative with respect to α —i.e., computing the Gâteaux derivative, for any suitable $h(\tau,t)$ we obtain:

$$\begin{aligned} \frac{d}{d\alpha} \mathcal{L}[\iota, f + \alpha h] \bigg|_{\alpha=0} &= \int_0^\infty e^{-\rho t} U'(c(t)) \left[-h(0,t) - \int_0^T \delta h(\tau,t)\, d\tau \right] dt \\ &\quad - \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial t} j(\tau,t)\, d\tau dt \\ &\quad + \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial \tau} j(\tau,t)\, d\tau dt. \end{aligned}$$

We employ integration by parts to show that:

$$\begin{aligned} \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial t} j(\tau,t)\, d\tau dt &= \int_0^T \int_0^\infty e^{-\rho t} \frac{\partial h}{\partial t} j(\tau,t)\, dt d\tau \\ &= \int_0^T \left(\lim_{s \to \infty} e^{-\rho s} h(\tau,s)\, j(\tau,s)] - h(\tau,0)\, j(\tau,0) \right) d\tau \\ &\quad - \int_0^T \int_0^\infty e^{-\rho t} \left(\frac{\partial j(\tau,t)}{\partial t} - \rho j(\tau,t) \right) h(\tau,t) dt d\tau, \end{aligned}$$

and

$$\int_0^\infty e^{-\rho t} \int_0^T \frac{\partial h}{\partial \tau} j(\tau,t)\, d\tau dt = \int_0^\infty e^{-\rho t} \left[h(T,t)\, j(T,t) - h(0,t)\, j(0,t) - \int_0^T h(\tau,t) \frac{\partial j}{\partial \tau} d\tau \right] dt.$$

Replacing these calculations in the Lagrangian, and equating it to zero, yields:

$$
\begin{aligned}
0 \;=\; & \int_0^\infty e^{-\rho t} U'(c(t)) \left[-h(0,t) - \int_0^T \delta h(\tau,t)\, d\tau \right] dt \\
& + \int_0^\infty \int_0^T e^{-\rho t} \left(-\rho j - \frac{\partial j}{\partial \tau} + \frac{\partial j}{\partial t} \right) h(\tau,t)\, d\tau dt \\
& + \int_0^\infty e^{-\rho t} \left(h(T,t)\, j(T,t) - h(0,t)\, j(0,t) \right) dt \\
& - \int_0^\infty \lim_{s\to\infty} e^{-\rho s} h(\tau,s)\, j(\tau,s)\, d\tau + h(\tau,0)\, j(\tau,0)\,.
\end{aligned}
$$

We rearrange terms to obtain:

$$
\begin{aligned}
0 \;=\; & -\int_0^\infty e^{-\rho t} \left[U'(c(t)) - j(0,t) \right] h(0,t)\, dt \\
& + \int_0^\infty \int_0^T e^{-\rho t} \left(-\rho j - U'(c)\, \delta - \frac{\partial j}{\partial \tau} + \frac{\partial j}{\partial t} \right) h(\tau,t)\, d\tau dt \\
& - \int_0^\infty e^{-\rho t} \left(h(T,t)\, j(T,t) \right) dt \\
& - \int_0^\infty \lim_{s\to\infty} e^{-\rho s} h(\tau,s)\, j(\tau,s)\, d\tau + h(\tau,0)\, j(\tau,0)\,.
\end{aligned}
\tag{C.1}
$$

Since $h(T,t) = h(\tau,0) = 0$ is a condition for any admissible variation, then, both the third line in equation (C.1) and the second term in the fourth line are equal to zero. Furthermore, because (C.1) needs to hold for any feasible variation $h(\tau,t)$, all the terms that multiply $h(\tau,t)$ should equal zero. The latter, yields a system of necessary conditions for the Lagrange multipliers:

$$
\begin{aligned}
\rho j(\tau,t) &= -\delta U'(c(t)) + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau}, \text{ if } \tau \in (0,T], \\
j(0,t) &= -U'(c(t))\,, \text{ if } \tau = 0, \\
\lim_{t\to\infty} e^{-\rho t} j(\tau,t) &= 0, \text{ if } \tau \in (0,T].
\end{aligned}
\tag{C.2}
$$

Next, we perturb the control. We proceed in a similar fashion:

$$
\begin{aligned}
\left. \tfrac{d}{d\alpha} \mathcal{L}\left[\iota + \alpha h,, f\right] \right|_{\alpha=0} = \; & \int_0^\infty e^{-\rho t} U'(c(t)) \left[\int_0^T \left(\frac{\partial q}{\partial \iota} \iota(\tau,t) + q(\tau,t,\iota) \right) h(\tau,t)\, d\tau \right] dt \\
& + \int_0^\infty \int_0^T e^{-\rho t} h(\tau,t)\, j(\tau,t)\, d\tau dt.
\end{aligned}
$$

Collecting terms and setting the Lagrangian to zero, we obtain:

$$
\int_0^\infty \int_0^T e^{-\rho t} \left[j(\tau,t) + U'(c(t)) \left(\frac{\partial q}{\partial \iota} \iota(\tau,t) + q(\tau,t,\iota) \right) \right] h(\tau,t)\, d\tau dt = 0.
$$

Thus, setting the term in parenthesis to zero, amounts to setting:

$$
U'(c(t)) \left(\frac{\partial q}{\partial \iota} \iota(\tau,t) + q(\tau,t,\iota) \right) = -j(\tau,t)\,. \tag{C.3}
$$

Next, we define the Lagrange multiplier in terms of goods:

$$v(\tau,t) = -j(\tau,t)/U'(c(t)). \tag{C.4}$$

Taking the derivative of $v(\tau,t)$ with respect to t and τ we can express the necessary conditions, (C.2) in terms of v. In particular, we transform the PDE in (C.2) into the summary equations in the Proposition. That is:

$$\begin{aligned}
\left(\rho - \frac{U''(c(t))\,c(t)}{U'(c(t))}\frac{\dot{c}(t)}{c(t)}\right) v(\tau,t) &= \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0,T], \\
v(0,t) &= 1, \text{ if } \tau = 0, \\
\lim_{t\to\infty} e^{-\rho t} v(\tau,t) &= 0, \text{ if } \tau \in (0,T];
\end{aligned}$$

and the first-order condition, (C.3), is now given by:

$$\frac{\partial q}{\partial \iota}\iota(\tau,t) + q(\tau,t,\iota) = v(\tau,t)$$

as we intended to show. □

C.2 Duality

Given a path of resources $y(t)$, the primal problem, the one solved in section 2, is given by:

$$V[f(\cdot,0)] = \max_{\{\iota(\tau,t),c(t)\}_{t\in[0,\infty),\tau\in[0,T]}} \int_t^\infty e^{-\rho(s-t)}u(c(s))ds \text{ s.t.}$$

$$c(t) = y(t) - f(0,t) + \int_0^T [q(\tau,t,\iota)\,\iota(\tau,t) - \delta f(\tau,t)]\,d\tau$$

$$\frac{\partial f}{\partial t} = \iota(\tau,t) + \frac{\partial f}{\partial \tau}; f(\tau,0) = f_0(\tau).$$

Here we show that this problem has a dual formulation. This dual formulation, minimizes the resources needed to sustain a given path of consumption $c(t)$:

$$D[f(\cdot,0)] = \min_{\{\iota(\tau,t)\}_{t\in[0,\infty),\tau\in[0,T]}} \int_0^\infty e^{-\int_0^t r(s)ds} y(t)dt \text{ s.t.}$$

$$c(t) = y(t) - f(0,t) + \int_0^T [q(\tau,t,\iota)\iota(\tau,t) - \delta f(\tau,t)]\,d\tau$$

$$\frac{\partial f}{\partial t} = \iota(\tau,t) + \frac{\partial f}{\partial \tau}; f(\tau,0) = f_0(\tau)$$

$$r(t) = \rho - \frac{U''(c(t))\,c(t)}{U'(c(t))}\frac{\dot{c}(t)}{c(t)}.$$

Proposition 5. *Consider the solution $\{c^*(t), \iota^*(\tau,t), f^*(\tau,t)\}_{t\geq 0,\tau\in(0,T]}$ to the Primal Problem given f_0. Then, given the path of consumption $c^*(t)$, $\{y^*(t), \iota^*(\tau,t), f^*(\tau,t)\}_{t\in[0,\infty),\tau\in(0,T]}$ solves the Dual Problem where:*

$$y^*(t) = c^*(t) + f^*(0,t) + \int_0^T [q(\tau,t,\iota^*)\iota^*(\tau,t) - \delta f^*(\tau,t)]\,d\tau.$$

Proof. Step 1. We start following the steps of Proposition 1. We construct the Lagrangian for the Dual Problem in the space $\left\|e^{-\rho t/2}g\left(\tau,t\right)\right\|^{2}<\infty$. After replacing the resources $y(t)$ needed to support a path of consumption $c(t)$ the budget constraint, is:

$$\begin{aligned}\mathcal{L}\left[\iota,f\right] &= \int_{0}^{\infty}e^{-\int_{0}^{t}r(t)ds}\left(c\left(t\right)+f\left(0,t\right)-\int_{0}^{T}\left[q\left(\tau,t,\iota\right)\iota\left(\tau,t\right)-\delta f\left(\tau,t\right)\right]d\tau\right)dt\\ &+\int_{0}^{\infty}\int_{0}^{T}e^{-\int_{0}^{t}r(t)ds}v\left(\tau,t\right)\left(-\frac{\partial f}{\partial t}+\iota\left(\tau,t\right)+\frac{\partial f}{\partial\tau}\right)d\tau dt,\end{aligned}$$

where $v\left(\tau,t\right)$ is the Lagrange multiplier associated to the law of motion of debt. We again consider a perturbation $h\left(\tau,t\right),e^{-\rho t}h\in L^{2}\left(\left[0,T\right]\times\left[0,\infty\right)\right)$, around the optimal solution. Recall that because f_0 is given, and $f\left(T,t\right)=0$, any feasible perturbation needs to meet: $h\left(\tau,0\right)=0$ and $h\left(T,t\right)=0$. At an optimal solution f, it must be the case that $\mathcal{L}\left[\iota,f\right]\geq\mathcal{L}\left[\iota,f+\alpha h\right]$ for any feasible perturbation $h(\tau,t)$. This implies that

$$\begin{aligned}\left.\frac{\partial}{\partial\alpha}\mathcal{L}\left[\iota,f+\alpha h\right]\right|_{\alpha=0} &= \int_{0}^{\infty}e^{-\int_{0}^{t}r(t)ds}\left[h\left(0,t\right)+\int_{0}^{T}\delta h\left(\tau,t\right)d\tau\right]dt\\ &-\int_{0}^{\infty}\int_{0}^{T}e^{-\int_{0}^{t}r(t)ds}\frac{\partial h}{\partial t}v\left(\tau,t\right)d\tau dt\\ &+\int_{0}^{\infty}\int_{0}^{T}e^{-\int_{0}^{t}r(t)ds}\frac{\partial h}{\partial\tau}v\left(\tau,t\right)d\tau dt.\end{aligned}$$

We again employ integration by parts to show that:

$$\begin{aligned}\int_{0}^{\infty}\int_{0}^{T}e^{-\int_{0}^{t}r(t)ds}\frac{\partial h}{\partial t}v\left(\tau,t\right)d\tau dt &= \int_{0}^{T}\int_{0}^{\infty}e^{-\int_{0}^{t}r(t)ds}v\left(\tau,t\right)\frac{\partial h}{\partial t}dtd\tau\\ &= \int_{0}^{T}\left(\lim_{s\to\infty}e^{-\int_{0}^{t}r(t)ds}h\left(\tau,s\right)v\left(\tau,s\right)]-h\left(\tau,0\right)v\left(\tau,0\right)\right)d\tau\\ &-\int_{0}^{T}\int_{0}^{\infty}e^{-\int_{0}^{t}r(t)ds}\left(\frac{\partial v\left(\tau,t\right)}{\partial t}-r(t)v\left(\tau,t\right)\right)h(\tau,t)dtd\tau\\ &= \int_{0}^{T}\left(\lim_{s\to\infty}e^{-\int_{0}^{t}r(t)ds}h\left(\tau,s\right)v\left(\tau,s\right)-h\left(\tau,0\right)v\left(\tau,0\right)\right)d\tau-\\ &\int_{0}^{\infty}e^{-\int_{0}^{s}r(u)du}\int_{0}^{T}\left(\frac{\partial v\left(\tau,t\right)}{\partial t}-r(t)v\left(\tau,t\right)\right)h(\tau,t)d\tau dt,\end{aligned}$$

and

$$\int_{0}^{\infty}e^{-\int_{0}^{t}r(t)ds}\int_{0}^{T}\frac{\partial h}{\partial\tau}v\left(\tau,t\right)d\tau dt = \int_{0}^{\infty}e^{-\int_{0}^{t}r(t)ds}\left[h\left(T,t\right)v\left(T,t\right)-h\left(0,t\right)v\left(0,t\right)-\int_{0}^{T}h(\tau,t)\frac{\partial v}{\partial\tau}d\tau\right]dt.$$

Replacing these calculations in the Lagrangian, and equating it to zero, yields:

$$
\begin{aligned}
0 &= \int_0^\infty e^{-\int_0^t r(t)ds}\left[h\left(0,t\right)+\int_0^T \delta h\left(\tau,t\right)d\tau\right]dtt \\
&+\int_0^\infty\int_0^T e^{-\int_0^t r(t)ds}\left(-r(t)v-\frac{\partial v}{\partial\tau}+\frac{\partial v}{\partial t}\right)h\left(\tau,t\right)d\tau dt \\
&+\int_0^\infty e^{-\int_0^t r(t)ds}\left(h\left(T,t\right)v\left(T,t\right)-h\left(0,t\right)v\left(0,t\right)\right)dt \\
&-\int_0^\infty \lim_{s\to\infty} e^{-\int_0^s r(u)du}h\left(\tau,s\right)v\left(\tau,s\right)d\tau.
\end{aligned}
$$

Again, the previous equation needs to hold for any feasible variation $h(\tau,t)$, all the terms that multiply $h(\tau,t)$ should be equal to zero. The latter, yields a system of necessary conditions for the Lagrange multipliers, and substituting for the value of $r(t)$:

$$
\begin{aligned}
\left(\rho-\frac{U''\left(c\left(t\right)\right)c\left(t\right)}{U'\left(c\left(t\right)\right)}\frac{\dot{c}(t)}{c\left(t\right)}\right)v\left(\tau,t\right) &= \delta+\frac{\partial v}{\partial t}-\frac{\partial v}{\partial\tau},\ \text{if } \tau\in(0,T], \\
v\left(0,t\right) &= 1,\ \text{ if } \tau=0, \\
\lim_{t\to\infty} e^{-\rho t}v\left(\tau,t\right) &= 0,\ \text{if } \tau\in(0,T].
\end{aligned}
\tag{C.5}
$$

By proceeding in a similar fashion with the control we arrive to:

$$
\left(\frac{\partial q}{\partial\iota}\iota\left(\tau,t\right)+q\left(\tau,t,\iota\right)\right)=-v\left(\tau,t\right). \tag{C.6}
$$

Note that system of equation (C.5) to (C.6) plus the budget constraint, the law of motion of debt, and initial debt f_0, are precisely the conditions that characterize the solution of the primal problem. □

C.3 Asymptotic behavior

Here we formally prove the limit conditions that we discussed after Proposition 1. In particular, we provide a complete asymptotic characterization. The following Proposition provides a summary.

Proposition 6. *Assume that $\rho > r_{ss}^*$, there exists a steady state if and only if $\bar{\lambda} > \bar{\lambda}_o$ for some $\bar{\lambda}_o$. If instead, $\bar{\lambda} \le \bar{\lambda}_o$, there is no steady state but consumption converges asymptotically to zero. In particular, the asymptotic behavior is:*

Case 1 (High Liquidity Costs). *For liquidity costs above the threshold value $\bar{\lambda} > \bar{\lambda}_o$, variables converge to a steady state characterized by the following system:*

$$
\begin{aligned}
\frac{\dot{c}_{ss}}{c_{ss}} &= 0 \\
r_{ss} &= 0 && \text{(C.7)} \\
\iota_{ss}(\tau) &= \frac{\psi_{ss}(\tau)-v_{ss}(\tau)}{\bar{\lambda}\psi_{ss}\left(\tau\right)}, && \text{(C.8)} \\
v_{ss}(\tau) &= \frac{\delta}{\rho}(1-e^{-\rho\tau})+e^{-\rho\tau} && \text{(C.9)} \\
f_{ss}(\tau) &= \int_\tau^T \iota_{ss}(s)ds && \text{(C.10)} \\
c_{ss} &= y_{ss}-f_{ss}(0)+\int_0^T\left[\psi_{ss}(\tau)\iota_{ss}(\tau)-\frac{\bar{\lambda}\psi_{ss}\left(\tau\right)}{2}\iota_{ss}(\tau)^2-\delta f_{ss}(\tau)\right]. && \text{(C.11)}
\end{aligned}
$$

Case 2 (Low Liquidity Costs). *For liquidity costs below the threshold value* $0 < \bar{\lambda} \leq \bar{\lambda}_o$*, variables converge asymptotically to:*

$$\begin{aligned}
\lim{}_{s\to\infty}\frac{c(s)}{c(t)} &= e^{-\frac{(\rho-r_\infty(\bar{\lambda}))(s-t)}{\sigma}} \\
v_\infty(\tau, r_\infty(\bar{\lambda})) &= \frac{\delta}{r_\infty(\bar{\lambda})}(1-e^{-r_\infty(\bar{\lambda})\tau}) + e^{-r_\infty(\bar{\lambda})\tau} \\
\iota_\infty(\tau, r_\infty(\bar{\lambda})) &= \frac{\psi_{ss}(\tau) - v_\infty(\tau, r_\infty(\bar{\lambda}))}{\bar{\lambda}\psi_{ss}(\tau)} \\
f_\infty(\tau, r_\infty(\bar{\lambda})) &= \int_\tau^T \iota_\infty(s, r_\infty(\bar{\lambda}))ds
\end{aligned}$$

where $r_\infty(\bar{\lambda})$ *satisfies* $r^*_{ss} \leq r_\infty(\bar{\lambda}) < \rho$ *and solves:*

$$\begin{aligned}
c_\infty &= 0 \\
&= y_{ss} - f_\infty(0, r_\infty(\bar{\lambda})) + \int_0^T \left[\iota_\infty(\tau, r_\infty(\bar{\lambda}))\psi(\tau) - \frac{\bar{\lambda}\psi_{ss}(\tau,t)}{2}\iota_\infty(\tau, r_\infty(\bar{\lambda}))^2 - \delta f_\infty(\tau, r_\infty(\bar{\lambda}))\right]d\tau.
\end{aligned}$$

Threshold. *The threshold* $\bar{\lambda}_o$ *solves* $|c_{ss}|_{\bar{\lambda}=\bar{\lambda}_o} = 0$ *in (C.11) and* $\lim_{\bar{\lambda}\to\bar{\lambda}_o} r_\infty(\bar{\lambda}) = \rho$*.*

Proof. Step 1. First observe that as $\bar{\lambda} \to \infty$, the optimal issuance policy (2.12) approaches $\iota(\tau, t) = 0$. Thus, for that limit, $c_{ss} = y > 0$ and $f_{ss}(\tau) = 0$.

Step 2. Next, consider the system in Case 1 of Proposition 6 as a guess of a solution. Note that equations (C.8) to (C.11) meet the necessary conditions of Proposition 1 as long as $r(t) = \rho$. This because: $\iota_{ss}(\tau)$ meets the first order condition with respect to the control; $v_{ss}(\tau)$ solves the PDE for valuations; given $\iota_{ss}(\tau)$ and $v_{ss}(\tau)$ the stock of debt solves the KFE, thus, is given by $\int_\tau^T \iota_{ss}(s)ds$; and consumption is pinned down by the budget constraint. In addition, by construction, consumption determined in (C.11) does not depend on time; i.e. $\dot{c}(t) = 0$ and this implies that

$$r_{ss} \equiv r(t) = \rho.$$

Thus, the only thing we need to check is that there exists some $\bar{\lambda}$ finite such that consumption is positive.

Step 3. The system in equations (C.8) to (C.11) is continuous in $\bar{\lambda}$. Therefore, because $c_{ss} = y > 0$ for $\bar{\lambda} \to \infty$, there exists a value of $\bar{\lambda}$ such that the implied consumption by equations (C.8) to (C.11) is positive.

Step 4. We now prove that there is an interval where this solution holds. In particular, we will show that c_{ss} decreases as $\bar{\lambda}$ increases. Observe that, steady state internal valuations $v_{ss}(\tau)$ in (C.9) and bond prices $\psi(\tau)$ are independent of $\bar{\lambda}$. Steady-state debt issuance's $\iota_{ss}(\tau)$ in (C.8) are a monotonously decreasing function of $\bar{\lambda}$, because

$$\frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} = -\frac{1}{\bar{\lambda}}\iota_{ss}(\tau) < 0,$$

and therefore the total amount of debt at each maturity $f_{ss}(\tau)$ in (C.10) is also decreasing with $\bar{\lambda}$, because

$$\frac{\partial f_{ss}(\tau)}{\partial \bar{\lambda}} = -\frac{1}{\bar{\lambda}}f_{ss}(\tau) < 0.$$

If we take derivatives with respect to $\bar{\lambda}$ in the budget constraint (C.11) we obtain:

$$\begin{aligned}
\frac{\partial c_{ss}}{\partial \bar{\lambda}} &= - \quad \frac{\partial f_{ss}(0)}{\partial \bar{\lambda}} + \int_0^T \left[\psi_{ss}(\tau) \frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} - \frac{\psi_{ss}(\tau)}{2} \iota_{ss}(\tau)^2 - \bar{\lambda} \psi_{ss}(\tau)\, \iota_{ss}(\tau) \frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} - \delta \frac{\partial f_{ss}(\tau)}{\partial \bar{\lambda}} \right] d\tau \\
&= \quad \frac{1}{\bar{\lambda}} f_{ss}(0) - \frac{1}{\bar{\lambda}} \int_0^T \left[\psi_{ss}(\tau) \iota_{ss}(\tau) + \bar{\lambda} \frac{\psi_{ss}(\tau)}{2} \iota_{ss}(\tau)^2 - \bar{\lambda} \psi_{ss}(\tau)\, \iota_{ss}(\tau)^2 - \delta f_{ss}(\tau) \right] d\tau \\
&= \quad -\frac{1}{\bar{\lambda}} c_{ss} < 0.
\end{aligned}$$

Observe that $\iota_{ss}(\tau)$ can be made arbitrarily small by increasing $\bar{\lambda}$. Thus, there exists a value of $\bar{\lambda} \geq 0$ such that $c_{ss} = 0$ in the system above. We denote this value by $\bar{\lambda}_o$.

Step 5. For $\bar{\lambda} \leq \bar{\lambda}_o$, if a steady state existed, it would imply $c_{ss} < 0$, outside of the range of admissible values. Therefore, there is no steady state in this case. Assume that the economy grows asymptotically at rate $g_\infty(\bar{\lambda}) \equiv \lim_{t\to\infty} \frac{1}{c(t)} \frac{dc}{dt}$. If $g_\infty(\bar{\lambda}) > 0$ then consumption would grow to infinity, which violates the budget constraint. Thus, if there exists an asymptotic the growth rate, it is negative: $g_\infty(\bar{\lambda}) < 0$. If we define $r_\infty(\bar{\lambda})$ as

$$r_\infty(\bar{\lambda}) \equiv \left(\rho + \sigma g(\bar{\lambda})\right) < \rho,$$

the growth rate of the economy can be expressed as

$$g_\infty(\bar{\lambda}) = -\frac{\left(\rho - r_\infty(\bar{\lambda})\right)}{\sigma}.$$

When this is the case, the asymptotic valuation is

$$v_\infty\left(\tau, r_\infty(\bar{\lambda})\right) = \frac{\delta\left(1 - e^{-r_\infty(\bar{\lambda})\tau}\right)}{r_\infty(\bar{\lambda})} + e^{-r_\infty(\bar{\lambda})\tau}.$$

To obtain the discount factor bounds, observe that if $v_\infty\left(\tau, r_\infty(\bar{\lambda})\right) \leq \psi_{ss}(\tau)$ the optimal issuance is non-negative. Otherwise issuances would be negative at all maturities and the country would be an asymptotic net asset holder. This cannot be an optimal solution as this implies that consumption can be increased just by reducing the amount of foreign assets. Therefore, $r_\infty(\bar{\lambda}) \geq r^*$. Finally, by definition $r_\infty(\bar{\lambda}) < \rho$. □

C.4 No liquidity costs: $\bar{\lambda} = 0$

Proposition 7. *(Optimal Policy with Liquid Debt) Assume that* $\lambda(\tau, t, \iota) = 0$. *If a solution exists, then consumption satisfies equation (2.11) with* $r^*(t) = r(t)$ *and the initial condition* $B(0) = \int_0^\infty \exp\left(-\int_0^s r^*(u)du\right)\left(c(s) - y(s)\right) ds$. *Given the optimal path of consumption, any solution* $\iota(\tau, t)$ *consistent with (2.1), (2.17) and*

$$\dot{B}(t) = r^*(t)B(t) + c(t) - y(t), \text{ for } t > 0, \tag{C.12}$$

is an optimal solution.

Proof. Step 1. The first part of the proof is just a direct consequence of the first-order condition $v(\tau, t) = \psi(\tau, t)$ for bond issuance. Bond prices are given by (2.4) while the government valuations are given by (2.10). Since both equations must be equal in a bounded solution, we conclude that

$$r^*(t) = r(t) = \rho - \frac{U''(c(t))}{U'(c(t))} \frac{dc}{dt},$$

must describe the dynamics of consumption.

Step 2. The second part of the proof derives the law of motion of $B(t)$. First we take the derivative with respect to time at both sides of definition (2.17), that we repeat for completion:

$$B\left(t\right)=\int_{0}^{T}\psi(\tau,t)f\left(\tau,t\right)d\tau.$$

Recall that, from the law of motion of debt, equation (2.1), it holds that:

$$\iota\left(\tau,t\right)=-\frac{\partial f}{\partial t}+\frac{\partial f}{\partial\tau}.$$

To express the budget constraint in terms of f, we substitute $\iota\left(\tau,t\right)$ into the budget constraint:

$$c\left(t\right)=y\left(t\right)-f\left(0,t\right)+\int_{0}^{T}\left[\psi(\tau,t)\left(\frac{\partial f}{\partial t}-\frac{\partial f}{\partial\tau}\right)-\delta f\left(\tau,t\right)\right]d\tau. \tag{C.13}$$

We would like to rewrite equation (C.13). Therefore, first, we apply integration by parts to the following expression:

$$\int_{0}^{T}\psi(\tau,t)\frac{\partial f}{\partial\tau}d\tau=\psi(T,t)f\left(T,t\right)-\psi(0,t)f\left(0,t\right)-\int_{0}^{T}\frac{\partial\psi}{\partial\tau}f\left(\tau,t\right)d\tau.$$

As long as the solution is smooth, it holds that $f\left(T,t\right)=0$. Further, recall that by construction $\psi(0,t)=1$. Hence:

$$\int_{0}^{T}\psi(\tau,t)\frac{\partial f}{\partial\tau}d\tau=-f\left(0,t\right)-\int_{0}^{T}\frac{\partial\psi}{\partial\tau}f\left(\tau,t\right)d\tau. \tag{C.14}$$

Second, from the pricing equation of international investors, we know that

$$\frac{\partial\psi}{\partial\tau}=-r^{*}\left(t\right)\psi(\tau,t)+\delta+\frac{\partial\psi}{\partial t}.$$

Then, we obtain:

$$\int_{0}^{T}\psi(\tau,t)\frac{\partial f}{\partial\tau}d\tau=-f\left(0,t\right)-\int_{0}^{T}\left[\delta+\psi_{t}(\tau,t)-r(t)\psi(\tau,t)\right]f\left(\tau,t\right)d\tau. \tag{C.15}$$

We substitute (C.14) and (C.15) into (C.13), and thus:

$$\begin{aligned}c(t)=&y\left(t\right)-f\left(0,t\right)+\int_{0}^{T}\left[\psi(\tau,t)\frac{\partial f}{\partial t}-\delta f\left(\tau,t\right)\right]d\tau...\\&-\left\{-f\left(0,t\right)-\int_{0}^{T}\left[\delta+\psi_{t}(\tau,t)-r(t)\psi(\tau,t)\right]f\left(\tau,t\right)d\tau\right\}\\=&y\left(t\right)+\int_{0}^{T}\left[\psi(\tau,t)f_{t}\left(\tau,t\right)+\psi_{t}(\tau,t)f\left(\tau,t\right)\right]d\tau-\int_{0}^{T}r^{*}(t)\psi(\tau,t)f\left(\tau,t\right)d\tau.\end{aligned}$$

Rearranging terms and employing the definitions above, we obtain:

$$\dot{B}\left(t\right)=c\left(t\right)-y(t)+r^{*}(t)B(t),$$

as desired. $\square$

C.5 Limiting distribution: $\bar{\lambda} \to 0$

Proposition 8. *(Limiting distribution) In the limit as liquidity costs vanish, $\bar{\lambda} \to 0$, the asymptotic optimal issuance is given by*

$$\iota_{\infty}^{\bar{\lambda}\to 0}(\tau) = \lim_{\bar{\lambda}\to 0} \iota_{\infty}(\tau) = \frac{1 + \left[-1 + (r^*/\delta - 1)\, r_{ss}^* \tau\right] e^{-r_{ss}^* \tau}}{1 + \left[-1 + (r^*/\delta - 1)\, r_{ss}^* T\right] e^{-r_{ss}^* T}} \frac{\psi_{ss}(T)}{\psi_{ss}(\tau)} \kappa, \tag{C.16}$$

where constant $\kappa > 0$ is such that $y_{ss} - f_{\infty}^{\bar{\lambda}\to 0}(0) + \int_0^T \left[\iota_{\infty}^{\bar{\lambda}\to 0}(\tau)\, \psi_{ss}(\tau) - \delta f_{\infty}^{\bar{\lambda}\to 0}(\tau)\right] d\tau = 0$, and $f_{\infty}^{\bar{\lambda}\to 0}(\tau) = \int_{\tau}^{T} \iota_{\infty}^{\bar{\lambda}\to 0}(s)\, ds$.

Proof. Consider the following limit:

$$\begin{aligned}
\iota_{\infty}^{\bar{\lambda}\to 0}(\tau) &\equiv \lim_{\bar{\lambda}\to 0} \iota_{\infty}(\tau, r_{\infty}(\bar{\lambda})) \\
&= \lim_{\bar{\lambda}\to 0} \frac{\psi_{ss}(\tau) - v_{\infty}(\tau, r_{\infty}(\bar{\lambda}))}{\bar{\lambda}\psi_{ss}(\tau)} \\
&= \lim_{\bar{\lambda}\to 0} \frac{1}{\bar{\lambda}\psi(\tau)} \left[\frac{\delta\left(1 - e^{-r_{ss}^*\tau}\right)}{r_{ss}^*} - \frac{\delta\left(1 - e^{-r_{\infty}(\bar{\lambda})\tau}\right)}{r_{\infty}(\bar{\lambda})} + e^{-r_{ss}^*\tau} - e^{-r_{\infty}(\bar{\lambda})\tau} \right].
\end{aligned}$$

This is a limit of the form $\frac{0}{0}$ as $\lim_{\bar{\lambda}\to 0} r_{\infty}(\bar{\lambda}) = r^*$.[29] We do not have an expression for $r_{\infty}(\bar{\lambda})$, so we cannot apply L'Hôpital's rule directly. Instead, we compute:

$$\lim_{\bar{\lambda}\to 0} \frac{\iota_{\infty}(\tau, r_{\infty}(\bar{\lambda}))}{\iota_{\infty}(T, r_{\infty}(\bar{\lambda}))} = \lim_{r_{\infty}(\bar{\lambda})\to r^*} \frac{\frac{\delta\left(1-e^{-r^*\tau}\right)}{r^*} - \frac{\delta\left(1-e^{-r_{\infty}(\bar{\lambda})\tau}\right)}{r} + e^{-r^*\tau} - e^{-r_{\infty}(\bar{\lambda})\tau}}{\frac{\delta\left(1-e^{-r^*T}\right)}{r^*} - \frac{\delta\left(1-e^{-r_{\infty}(\bar{\lambda})T}\right)}{r} + e^{-r^*T} - e^{-r_{\infty}(\bar{\lambda})T}} \frac{\psi(T)}{\psi(\tau)},$$

which also has a limit of the form $\frac{0}{0}$. Now we can apply L'Hôpital's. We obtain:

$$\begin{aligned}
\lim_{\bar{\lambda}\to 0} \frac{\iota_{\infty}(\tau, r_{\infty}(\bar{\lambda}))}{\iota_{\infty}(T, r_{\infty}(\bar{\lambda}))} &= \frac{\frac{-\delta r^*\tau e^{-r^*\tau} + \delta\left(1-e^{-r^*\tau}\right)}{r^{*2}} + \tau e^{-r^*\tau}}{\frac{-\delta r^* T e^{-r^*T} + \delta\left(1-e^{-r^*T}\right)}{r^{*2}} + T e^{-rT}} \frac{\psi(T)}{\psi(\tau)} \\
&= \frac{1 + \left[-1 + (r^*/\delta - 1)\, r^*\tau\right] e^{-r^*\tau}}{1 + \left[-1 + (r^*/\delta - 1)\, r^* T\right] e^{-r^*T}} \frac{\psi(T)}{\psi(\tau)}.
\end{aligned}$$

If we define

$$\kappa \equiv \lim_{\bar{\lambda}\to 0} \iota_{\infty}(T, r_{\infty}(\bar{\lambda}))$$

then

$$\lim_{\bar{\lambda}\to 0} \iota_{\infty}(\tau, r_{\infty}(\bar{\lambda})) = \frac{1 + \left[-1 + (r^*/\delta - 1)\, r^*\tau\right] e^{-r^*\tau}}{1 + \left[-1 + (r^*/\delta - 1)\, r^* T\right] e^{-r^*T}} \frac{\psi(T)}{\psi(\tau)} \kappa.$$

The value of κ then must be consistent with zero consumption:

$$y_{ss} - f_{\infty}^{\bar{\lambda}\to 0}(0) + \int_0^T \left[\iota_{\infty}^{\bar{\lambda}\to 0}(\tau)\, \psi_{ss}(\tau) - \delta f_{\infty}^{\bar{\lambda}\to 0}(\tau)\right] d\tau = 0,$$

for $f_{\infty}^{\bar{\lambda}\to 0}(\tau) = \int_{\tau}^{T} \iota_{\infty}^{\bar{\lambda}\to 0}(s)\, ds$. $\square$

[29] We drop the sub-index ss to ease the notation.

C.6 Proof of Proposition 2

We need first the following lemma:

Lemma 1. *Given a fixed t, the Gâteaux derivative of the post-shock value functional $V[f(\cdot,t)]$, defined by equation (2.6), with respect to the debt distribution $f(\cdot,t)$ is the valuation $j(\tau,t) = -U'(c(t))\, v(\tau,t)$* satisfying equation (2.8):

$$\frac{d}{d\alpha} V[f(\tau,t) + \alpha h(\tau,t)]\bigg|_{\alpha=0} = \int_0^T j(\tau,t)\, h(\tau,t)\, d\tau.$$

Proof. To simplify notation assume, without loss of generality, that $t = 0$. To avoid confusions, we denote by (ι^*, f^*) the optimal issuance policy and debt distributions. First, note that

$$V[f(\cdot,0)] = \mathcal{L}[\iota^*, f^*].$$

This follows from the fact that

$$\begin{aligned}
V[f(\cdot,0)] &= \int_0^\infty e^{-\rho t} U\left(y(t) - f^*(0,t) + \int_0^T [q(\tau,t,\iota^*)\,\iota^*(\tau,t) - \delta f^*(\tau,t)]\, d\tau\right) dt \\
&= \int_0^\infty e^{-\rho t} U\left(y(t) - f^*(0,t) + \int_0^T [q(\tau,t,\iota^*)\,\iota^*(\tau,t) - \delta f^*(\tau,t)]\, d\tau\right) dt \\
&+ \int_0^\infty \int_0^T e^{-\rho t} j^*(\tau,t) \left(-\frac{\partial f^*}{\partial t} + \iota^*(\tau,t) + \frac{\partial f^*}{\partial \tau}\right) d\tau dt, \\
&= \mathcal{L}[\iota^*, f^*].
\end{aligned}$$

The first line is the definition of $V[f(\cdot,0)]$, the second line is the fact that

$$-\frac{\partial f^*}{\partial t} + \iota^*(\tau,t) + \frac{\partial f^*}{\partial \tau} = 0$$

for for every τ, t and the last line is the definition of the Lagrangian. Next we compute $\frac{d}{d\alpha}\mathcal{L}[\iota^*, f^* + \alpha h(\tau,0)]\Big|_{\alpha=0}$. The derivative with respect to a general variation $h(\tau,t)$ is given by:

$$\begin{aligned}
\frac{d}{d\alpha}\mathcal{L}[\iota, f + \alpha h]\bigg|_{\alpha=0} \quad &= \quad \int_0^\infty e^{-\rho t} U'(c(t)) \left[-h(0,t) - \int_0^T \delta h(\tau,t)\, d\tau\right] dt \\
&- \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial t} j(\tau,t)\, d\tau dt \\
&+ \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial \tau} j(\tau,t)\, d\tau dt.
\end{aligned} \tag{C.17}$$

We employ integration by parts to show that:

$$\begin{aligned}
\int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial t} j(\tau,t)\, d\tau dt &= \int_0^T \left(\lim_{s\to\infty} e^{-\rho s}[h(\tau,s)\, j(\tau,s)] - h(\tau,0)\, j(\tau,0)\right) d\tau \\
&- \int_0^T \int_0^\infty e^{-\rho t} \left(\frac{\partial j(\tau,t)}{\partial t} - \rho j(\tau,t)\right) h(\tau,t) dt d\tau
\end{aligned}$$

and

$$\int_0^\infty e^{-\rho t} \int_0^T \frac{\partial h}{\partial \tau} j(\tau,t)\, d\tau dt \quad = \quad \int_0^\infty e^{-\rho t} \left[h(T,t)\, j(T,t) - h(0,t)\, j(0,t) - \int_0^T h(\tau,t) \frac{\partial j}{\partial \tau} d\tau\right] dt.$$

Note that we have not yet used optimality. Using the particular case of interest, i.e.:

$$h(\tau,0) = h(\tau,t)\delta(t),$$

where $\delta(t)$ is the Dirac delta, and plugging it in equation (C.17):

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{L}\left[\iota^*, f^* + \alpha h(\cdot,0)\right]\Big|_{\alpha=0} &= U'\left(c^*(0)\right)\left[-h(0,0) - \int_0^T \delta h\left(\tau,0\right)d\tau\right] \\ &+ \int_0^T h\left(\tau,0\right) j\left(\tau,0\right)d\tau \\ &+ \int_0^T \left(\frac{\partial j\left(\tau,0\right)}{\partial t} - \rho j\left(\tau,0\right)\right) h(\tau,0)d\tau, \\ &- h\left(0,0\right) j\left(0,0\right) - \int_0^T h(\tau,0)\frac{\partial j}{\partial \tau}d\tau dt.\end{aligned}$$

Because (ι^*, f^*) is an optimum, we know that for all $\tau \in (0,T]$ the following holds:

$$\frac{\partial j\left(\tau,0\right)}{\partial t} - \rho j\left(\tau,0\right) - \frac{\partial j\left(\tau,0\right)}{\partial \tau} - \delta = 0,$$

and for $\tau = 0$ it also holds that $U'\left(c^*(0)\right) = -j\left(0,0\right)$. This implies that

$$\frac{d}{d\alpha}\mathcal{L}\left[\iota^*, f^* + \alpha u(\cdot,0)\right]\Big|_{\alpha=0} = \int_0^T h\left(\tau,0\right) j\left(\tau,0\right)d\tau.$$

Thus,

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{L}\left[\iota^*, f^* + \alpha h(\cdot,0)\right]\Big|_{\alpha=0} &= \int_0^T h\left(\tau,0\right) j\left(\tau,0\right)d\tau \\ &= \frac{d}{d\alpha}V\left[f(\cdot,0) + \alpha h(\cdot,0)\right]\Big|_{\alpha=0}.\end{aligned}$$

□

Proof. Proposition 2. Then we can proceed with the proof of Proposition 2. The Lagrangian is:

$$\begin{aligned}\mathcal{L}\left[\hat{\iota}, \hat{f}\right] = \mathbb{E}^{t^o}\Bigg[&\int_0^{t^o} e^{-\rho s}U\left(\hat{c}(s)\right)ds + e^{-\rho t^o}\mathbb{E}^X_{t^o}\left\{V\left[\hat{f}\left(\cdot,t^o\right), X\left(t^o\right)\right]\right\} \\ &+ \int_0^{t^o}\int_0^T e^{-\rho s}\hat{\jmath}\left(\tau,s\right)\left(-\frac{\partial \hat{f}}{\partial s} + \hat{\iota}\left(\tau,s\right) + \frac{\partial \hat{f}}{\partial \tau}\right)d\tau ds\Bigg]\end{aligned}$$

where $\mathbb{E}^{t^o}$ denotes the expectation with respect to the random time t^o. In this case $\hat{\jmath}\left(\tau,s\right)$ is the Lagrange multiplier associated to the law of motion of debt, before the shock.

Step 1. Re-writing the Lagrangian. Proceeding as in the proof of the risk-less case, as an intermediate step we integrate by parts the terms that involve time or maturity derivatives of $\hat{f}$. The Lagrangian $\mathcal{L}\left[\hat{\iota}, \hat{f}\right]$ can thus be expressed as:

$$\begin{aligned}\mathcal{L}\left[\hat{\iota},\hat{f}\right] &= \mathbb{E}^{t^o}\left[\int_0^{t^o} e^{-\rho s}U\left(\hat{c}(s)\right)ds + e^{-\rho t^o}V\left[\hat{f}\left(\cdot,t^o\right),X\left(t^o\right)\right]\right.\\ &\quad -\int_0^T e^{-\rho t^o}\hat{f}\left(\tau,t^o\right)\hat{\jmath}\left(\tau,t^o\right)d\tau + \int_0^T \hat{f}\left(\tau,0\right)\hat{\jmath}\left(\tau,0\right)d\tau\\ &\quad +\int_0^{t^o}\int_0^T e^{-\rho s}\hat{f}\left(\tau,s\right)\left(\frac{\partial\hat{\jmath}}{\partial s}-\rho\hat{\jmath}\left(\tau,s\right)\right)d\tau ds\\ &\quad +\int_0^{t^o} e^{-\rho s}\hat{f}\left(T,s\right)\hat{\jmath}\left(T,s\right)ds - \int_0^{t^o} e^{-\rho s}\hat{f}\left(0,s\right)\hat{\jmath}\left(0,s\right)ds\\ &\quad -\int_0^{t^o}\int_0^T e^{-\rho s}\hat{f}\left(\tau,s\right)\frac{\partial\hat{\jmath}}{\partial\tau}d\tau ds\\ &\quad +\int_0^{t^o}\int_0^T e^{-\rho s}\hat{\jmath}\left(\tau,s\right)\hat{\iota}\left(\tau,s\right)d\tau ds.\end{aligned}$$

If we group terms, substitute the terminal conditions $\hat{f}\left(T,s\right)=0$ and compute the expected value with respect to t^o, we can express the Lagrangian $\mathcal{L}\left[\hat{\iota},\hat{f}\right]$ as:

$$\begin{aligned}\mathcal{L}\left[\hat{\iota},\hat{f}\right] &= \int_0^\infty e^{-(\rho+\phi)s}U\left(\hat{c}(s)\right)ds\\ &\quad +\int_0^\infty e^{-(\rho+\phi)s}\phi V\left[\hat{f}\left(\cdot,s\right)\right]ds\\ &\quad -\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi\hat{f}\left(\tau,s\right)\hat{\jmath}\left(\tau,s\right)d\tau ds\\ &\quad +\int_0^T \hat{f}\left(\tau,0\right)\hat{\jmath}\left(\tau,0\right)d\tau\\ &\quad +\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{f}\left(\tau,s\right)\left(\frac{\partial\hat{\jmath}}{\partial s}-\rho\hat{\jmath}\left(\tau,s\right)\right)d\tau ds\\ &\quad -\int_0^\infty e^{-(\rho+\phi)s}\hat{f}\left(0,s\right)\hat{\jmath}\left(0,s\right)ds\\ &\quad -\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{f}\left(\tau,s\right)\frac{\partial\hat{\jmath}}{\partial\tau}d\tau ds\\ &\quad +\int_0^{t^o}\int_0^T e^{-(\rho+\phi)s}\hat{\jmath}\left(\tau,s\right)\hat{\iota}\left(\tau,s\right)d\tau ds.\end{aligned}$$

Step 2: Gâteaux derivatives. Next, we compute the Gâteaux derivatives with respect to each of the two arguments of the Lagrangian at a time. *Step 2.1: Gateaux derivative with respect to issuances.* We consider a perturbation around optimal issuances. Equalizing the Gâteaux derivative with respect to issuances to zero, i.e. $\mathcal{L}\left[\hat{\iota}+\alpha h,\hat{f}\right]\Big|_{\alpha=0}=0$, the result is identical to the riskless case:

$$U'\left(\hat{c}\left(t\right)\right)\left(\frac{\partial q}{\partial\hat{\iota}}\hat{\iota}\left(\tau,t\right)+q\left(\tau,t,\hat{\iota}\right)\right)=-\hat{\jmath}\left(\tau,t\right).$$

Step 2.2: Gateaux derivative of V with respect to the debt density. The Gâteaux derivative of the continuation value with respect to the debt density is

$$\frac{d}{d\alpha}V\left[\hat{f}\left(\cdot,s\right)+\alpha h\left(\cdot,s\right),X\left(s\right)\right]\Big|_{\alpha=0}=\mathbb{E}_s^X\left\{\int_0^T j\left(\tau,s\right)h\left(\tau,s\right)d\tau\right\}.$$

where we have applied Lemma 1. *Step 2.3: Gateaux derivative of the Lagrangian with respect to the debt density.* Since the distribution at the beginning $\hat{f}(\tau, 0)$ is given, any feasible perturbation must feature $h(\tau, 0) = 0$ for any $\tau \in (0, T]$. In addition, we know that $h(T, t) = 0$, because $\hat{f}(T, t) = 0$. The Gâteaux derivative of the Lagrangian with respect to the debt density is:

$$
\begin{aligned}
\frac{d}{d\alpha} \mathcal{L}\left[\hat{\iota}, \hat{f} + \alpha h\right]\Big|_{\alpha=0} = & \int_0^\infty e^{-(\rho+\phi)s} U'(\hat{c}(s)) \left[-h(0, s) + \int_0^T (-\delta) h(\tau, s) d\tau\right] ds \\
& + \int_0^\infty \int_0^T e^{-(\rho+\phi)s} \phi \mathbb{E}_s^X \left[j(\tau, s) h(\tau, s)\right] d\tau ds \\
& - \int_0^\infty \int_0^T e^{-(\rho+\phi)s} \theta h(\tau, s) \hat{j}(\tau, s) d\tau ds \\
& + \int_0^T h(\tau, 0) \hat{j}(\tau, 0) d\tau \\
& + \int_0^\infty \int_0^T e^{-(\rho+\phi)s} h(\tau, s) \left(\frac{\partial \hat{j}}{\partial s} - \rho \hat{j}(\tau, s)\right) ds d\tau \\
& - \int_0^\infty e^{-(\rho+\phi)s} h(0, s) \hat{j}(0, s) ds \\
& - \int_0^\infty \int_0^T e^{-(\rho+\phi)s} h(\tau, s) \frac{\partial \hat{j}}{\partial \tau} d\tau ds.
\end{aligned}
$$

The value of the Gâteaux derivative of the Lagrangian for any perturbation must be zero. Thus, again a necessary condition is to have all terms that multiply any entry of $h(\tau, s)$ add up to zero. We summarize the necessary conditions into:

$$
\rho \hat{j}(\tau, s) = -\delta U'(\hat{c}(s)) + \frac{\partial \hat{j}}{\partial s} - \frac{\partial \hat{j}}{\partial \tau} + \phi \left[\mathbb{E}_s^X j(\tau, s) - \hat{j}(\tau, s)\right], \tag{C.18}
$$

$$
\hat{j}(0, s) = -U'(\hat{c}(s)). \tag{C.19}
$$

Step 3: From Lagrange multipliers to valuations. We now employ the definitions of $\hat{v}(\tau, s) = -\hat{j}(\tau, s) / U'(\hat{c}(s))$ and $v(\tau, s) = -j(\tau, s) / U'(c(s))$. Thus, we can express equations (C.18)-(C.19) as:

$$
\begin{aligned}
\hat{r}(s) \hat{v}(\tau, s) &= \delta + \frac{\partial \hat{v}}{\partial s} - \frac{\partial \hat{v}}{\partial \tau} + \phi \left[\mathbb{E}_s^X \frac{U'(c(s))}{U'(\hat{c}(s))} v(\tau, s) - \hat{v}(\tau, s)\right] \\
\hat{v}(0, s) &= 1.
\end{aligned}
$$

Therefore valuations can be expressed as:

$$
\begin{aligned}
\hat{v}(\tau, t) \quad = \quad & e^{-\int_t^{t+\tau} (\hat{r}(s) + \phi) ds} \\
& + \theta \int_t^{t+\tau} e^{-\int_t^s (\hat{r}(u) + \phi) du} \left(\delta + \mathbb{E}_s^X \left[\frac{U'(c(t+s))}{U'(\hat{c}(t+s))} v(\tau - s, t+s)\right]\right) ds,
\end{aligned}
$$

as we wanted to show. □

C.7 Proof of Proposition 3

Proof. Step 1. Setting the Lagrangian. Let $\mathbb{V}\left[\hat{f}\left(\cdot,t^o\right),X\left(t^o\right)\right]$ denote the expected value of the government, at the instant t^o where the option to default is available, but prior to the decision of default. This value equals:

$$\mathbb{E}_{t^o}^{X}\left[\underbrace{\Gamma\left(V\left[\hat{f}\left(\cdot,t^o\right)\right],X\left(t^o\right)\right)}_{\text{Default}}+\underbrace{\Theta\left(V\left[\hat{f}\left(\cdot,t^o\right),X\left(t^o\right)\right]\right)V\left[\hat{f}\left(\cdot,t^o\right),X\left(t^o\right)\right]}_{\text{No default}}\right],$$

where the first term in the expectation is the expected utility conditional on default given by $\Gamma\left(x\right)\equiv\int_x^\infty zd\Theta(z)$. The second term is the probability of no default time the perfect-foresight value. The Lagrangian is:

$$\begin{aligned}\mathcal{L}\left[\hat{\imath},\hat{f},\hat{\psi}\right] = \mathbb{E}^{t^o}&\left[\int_0^{t^o}e^{-\rho s}U\left(\hat{c}(s)\right)ds+e^{-\rho t^o}\mathbb{V}\left[\hat{f}\left(\cdot,t^o\right),X\left(t^o\right)\right]\right.\\&+\int_0^{t^o}\int_0^T e^{-\rho s}\hat{\jmath}\left(\tau,s\right)\left(-\frac{\partial\hat{f}}{\partial s}+\hat{\imath}\left(\tau,s\right)+\frac{\partial\hat{f}}{\partial\tau}\right)d\tau ds\\&\left.+\int_0^{t^o}\int_0^T e^{-\rho s}\hat{\mu}\left(\tau,s\right)\left(-\hat{r}^*\left(s\right)\hat{\psi}(\tau,s)+\delta+\frac{\partial\hat{\psi}}{\partial s}-\frac{\partial\hat{\psi}}{\partial\tau}\right)d\tau ds\right].\end{aligned}$$

In the Lagrangian, $\mathbb{E}^{t^o}$ denotes the conditional expectation with respect to the random time t^o. Here $\hat{\jmath}\left(\tau,s\right)$ and $\hat{\mu}\left(\tau,s\right)$ are the Lagrange multipliers. The first set of multipliers, $\hat{\jmath}\left(\tau,s\right)$, are associated with the law of motion of debt and appears also in previous sections. The second set of multipliers, $\hat{\mu}\left(\tau,s\right)$, are associated with the law of motion of bond prices. These terms appear because the government understands how its influence on the maturity profile affects the incentives to default, and hence impacts bond prices. This happens through the terminal condition:

$$\begin{aligned}\hat{\psi}(\tau,t^o) &= \mathbb{E}_{t^o}^{X}\left\{\Theta\left(V\left[\hat{f}\left(\cdot,t^o\right),X\left(t^o\right)\right]\right)\psi(\tau,t^o)\right\},\\\hat{\psi}(0,t) &= 1.\end{aligned}$$

The terminal condition reflects that, at date t^o, the bond price is zero if default occurs. Otherwise it equals the perfect-foresight price, $\psi(\tau,t^o)$, if default does not occur.

Step 1.2. Re-writing the Lagrangian. Proceeding as in the proof of the riskless case, as an intermediate step we integrate by parts the terms that involve time or maturity derivatives of $\hat{f}$ and $\hat{\psi}$. The Lagrangian $\mathcal{L}\left[\hat{\imath},\hat{f},\hat{\psi}\right]$ can thus be expressed as:

$$
\begin{aligned}
&\mathbb{E}^{t^o}\left[\int_0^{t^o} e^{-\rho s} U\left(\hat{c}(s)\right) ds + e^{-\rho t^o}\mathbb{V}\left[\hat{f}\left(\cdot, t^o\right), X\left(t^o\right)\right] - \int_0^T e^{-\rho t^o}\hat{f}\left(\tau, t^o\right)\hat{\jmath}\left(\tau, t^o\right) d\tau\right.\\
&+\int_0^T \hat{f}\left(\tau, 0\right)\hat{\jmath}\left(\tau, 0\right) d\tau + \int_0^{t^o}\int_0^T e^{-\rho s}\hat{f}\left(\tau, s\right)\left(\frac{\partial \hat{\jmath}}{\partial s} - \rho\hat{\jmath}\left(\tau, s\right)\right) ds d\tau\\
&+\int_0^{t^o} e^{-\rho s}\hat{f}\left(T, s\right)\hat{\jmath}\left(T, s\right) ds - \int_0^{t^o} e^{-\rho s}\hat{f}\left(0, s\right)\hat{\jmath}\left(0, s\right) ds\\
&-\int_0^{t^o}\int_0^T e^{-\rho s}\hat{f}\left(\tau, s\right)\frac{\partial \hat{\jmath}}{\partial \tau} d\tau ds + \int_0^{t^o}\int_0^T e^{-\rho s}\hat{\jmath}\left(\tau, s\right)\hat{\imath}\left(\tau, s\right) d\tau ds\\
&+\int_0^{t^o}\int_0^T e^{-\rho s}\hat{\mu}\left(\tau, s\right)\left(-\hat{r}^*\left(s\right)\hat{\psi}(\tau, s) + \delta\right) d\tau ds,\\
&+\int_0^T\left[e^{-\rho t^o}\hat{\mu}\left(\tau, t^o\right)\hat{\psi}\left(\tau, t^o\right) - \hat{\mu}\left(\tau, 0\right)\hat{\psi}\left(\tau, 0\right)\right] d\tau\\
&-\int_0^{t^o}\int_0^T e^{-\rho s}\hat{\psi}(\tau, s)\left(\frac{\partial \hat{\mu}}{\partial s} - \rho\hat{\mu}(\tau, s)\right) d\tau ds\\
&-\int_0^{t^o} e^{-\rho s}\left[\hat{\mu}\left(T, s\right)\hat{\psi}\left(T, s\right) - e^{-\rho s}\hat{\mu}\left(0, s\right)\hat{\psi}\left(0, s\right)\right] ds\\
&\left.+\int_0^{t^o}\int_0^T e^{-\rho s}\hat{\psi}\left(\tau, s\right)\frac{\partial \hat{\mu}}{\partial \tau} d\tau ds\right].
\end{aligned}
$$

Step 1.3. Computing expectations. If we group terms, substitute the terminal conditions $f\left(T, s\right) = 0$ and $\hat{\psi}(\tau, t^o) = \Theta\left(V\left[\hat{f}\left(\cdot, t^o\right), X\left(t^o\right)\right]\right)\psi(\tau, t^o)$ and compute the expected value with respect to t^o, we can express the Lagrangian $\mathcal{L}\left[\hat{\iota},\hat{f},\hat{\psi}\right]$ as:

$$
\begin{aligned}
&\int_0^\infty e^{-(\rho+\phi)s}U\left(\hat{c}(s)\right)ds \\
&-\int_0^\infty e^{-(\rho+\phi)s}\hat{f}\left(0,s\right)\hat{\jmath}\left(0,s\right)ds \\
&-\int_0^\infty e^{-(\rho+\phi)s}\left[\hat{\mu}\left(T,s\right)\hat{\psi}\left(T,s\right)-\mu\left(0,s\right)\hat{\psi}\left(0,s\right)\right]ds \\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{f}\left(\tau,s\right)\left(\frac{\partial\hat{\jmath}}{\partial s}-\rho\hat{\jmath}\left(\tau,s\right)\right)dsd\tau \\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{f}\left(\tau,s\right)\frac{\partial\hat{\jmath}}{\partial\tau}d\tau ds \\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{\jmath}\left(\tau,s\right)\hat{\iota}\left(\tau,s\right)d\tau ds \\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{\mu}\left(\tau,s\right)\left(-\hat{r}^*\left(s\right)\hat{\psi}(\tau,s)+\delta\right)d\tau ds \\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{\psi}(\tau,s)\left(\frac{\partial\hat{\mu}}{\partial s}-\rho\hat{\mu}(\tau,s)\right)d\tau ds \\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{\psi}\left(\tau,s\right)\frac{\partial\hat{\mu}}{\partial\tau}d\tau ds \\
&+\int_0^T f\left(\tau,0\right)\hat{\jmath}\left(\tau,0\right)d\tau-\int_0^T\hat{\mu}\left(\tau,0\right)\hat{\psi}\left(\tau,0\right)d\tau \\
&+\int_0^\infty e^{-(\rho+\phi)s}\phi\mathbb{V}\left[\hat{f}\left(\cdot,s\right),X(s)\right]ds \\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi\hat{f}\left(\tau,s\right)\hat{\jmath}\left(\tau,s\right)d\tau ds \\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi\hat{\mu}\left(\tau,s\right)\mathbb{E}_s^X\left\{\theta\left(V\left[\hat{f}\left(\cdot,s\right),X(s)\right]\right)\psi(\tau,s)\right\}d\tau ds.
\end{aligned}
$$

Next, we compute the Gâteaux derivatives with respect to each of the three arguments of the value function at a time.

Step 2. Computing the derivatives. Step 2.1. Gâteaux derivative with respect to the issuances. If we consider a perturbation around issuances and equalize it to zero, $\frac{\partial}{\partial\alpha}\mathcal{L}\left[\hat{\iota}+\alpha h,\hat{f},\hat{\psi}\right]\Big|_{\alpha=0}=0$, the result is identical to the risk-less case:

$$U'\left(\hat{c}\left(t\right)\right)\left(\frac{\partial q}{\partial\iota}\hat{\iota}\left(\tau,t\right)+q\left(t,\tau,\hat{\iota}\right)\right)=-\hat{\jmath}\left(\tau,t\right).$$

Step 2.2. Gâteaux derivative with respect to the debt density. Since the distribution at the beginning $f\left(\tau,0\right)$ is given, any feasible perturbation must feature $h(\tau,0)=0$ for any $\tau\in(0,T]$. In addition, we know that $h(T,t)=0$, because $f(T,t)=0$. The Gâteaux derivative of the continuation value with respect to the debt density is:

$$\frac{d}{d\alpha}\mathbb{V}\left[\hat{f}\left(\cdot,s\right)+\alpha h\left(\cdot,s\right),X\left(s\right)\right]\Big|_{\alpha=0}=\mathbb{E}_s^X\left\{\Theta\left(V\left[\hat{f}\left(\cdot,s\right),X\left(s\right)\right]\right)\int_0^T j\left(\tau,s\right)h\left(\tau,s\right)d\tau\right\},$$

where we have taken into account the fact that $\frac{d}{dx}\left(\Gamma\left(x\right)+\Theta\left(x\right)x\right)=\Theta\left(x\right)$ and −from the perfect foresight problem− :

$$\frac{d}{d\alpha}V\left[\hat{f}\left(\cdot,s\right)+\alpha h\left(\cdot,s\right)\right]\Big|_{\alpha=0}=\int_0^T j\left(\tau,s\right)h\left(\tau,s\right)d\tau.$$

Similarly, the Gâteaux derivative of the terminal bond price with respect to the debt density is

$$\frac{d}{d\alpha}\,\mathbb{E}_s^X\left\{\Theta\left(V\left[\hat{f}\left(\cdot,s\right)+\alpha h\left(\cdot,s\right),X\left(s\right)\right]\right)\psi(\tau,s)\right\}\Big|_{\alpha=0}=\dots$$
$$\mathbb{E}_s^X\left\{\theta\left(V\left[\hat{f}\left(\cdot,s\right),X\left(s\right)\right]\right)\psi(\tau,s)\int_0^T j\left(\tau',s\right)h\left(\tau',s\right)d\tau'\right\},$$

where $\theta(x)\equiv\frac{d}{dx}\Theta\left(x\right)$ is the probability density. The Gâteaux derivative of the Lagrangian with respect to the debt density, $\frac{d}{d\alpha}\,\mathcal{L}\left[\hat{\imath},\hat{f}+\alpha h,\hat{\psi}\right]\Big|_{\alpha=0}$, is thus:

$$\begin{aligned}
&\int_0^\infty e^{-(\rho+\phi)s}U'\left(\hat{c}\left(s\right)\right)\left[-h\left(0,s\right)+\int_0^T\left(-\delta\right)h\left(\tau,s\right)d\tau\right]ds\\
&-\int_0^\infty e^{-(\rho+\phi)s}h\left(0,s\right)\hat{\jmath}\left(0,s\right)ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}h\left(\tau,s\right)\left(\frac{\partial\hat{\jmath}}{\partial s}-\rho\hat{\jmath}\left(\tau,s\right)\right)dsd\tau\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}h\left(\tau,s\right)\frac{\partial\hat{\jmath}}{\partial\tau}d\tau ds\\
&+\int_0^T h\left(\tau,0\right)\hat{\jmath}\left(\tau,0\right)d\tau\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\theta\mathbb{E}_s^X\left\{\Theta\left(V\left[\hat{f}\left(\cdot,s\right),X\left(s\right)\right]\right)j\left(\tau,s\right)\right\}h\left(\tau,s\right)d\tau ds\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi h\left(\tau,s\right)\hat{\jmath}\left(\tau,s\right)d\tau ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi\mu\left(m,s\right)\mathbb{E}_s^X\left\{\theta\left(V\left[\hat{f}\left(\cdot,s\right),X\left(s\right)\right]\right)\psi(m,s)\int_0^T j\left(\tau,s\right)h\left(\tau,s\right)d\tau\right\}dmds.
\end{aligned}$$

The value of the Gâteaux derivative of the Lagrangian for any perturbation, must be zero, i.e. $\frac{d}{d\alpha}\,\mathcal{L}\left[\hat{\imath},\hat{f}+\alpha h,\hat{\psi}\right]\Big|_{\alpha=0}=0$. Thus, a necessary condition is that all terms that multiply any entry of $h\left(\tau,s\right)$ add up to zero. We summarize the necessary conditions into:

$$\begin{aligned}
\rho\hat{\jmath}\left(\tau,s\right)&=\left(-\delta\right)U'\left(\hat{c}\left(s\right)\right)+\frac{\partial\hat{\jmath}}{\partial s}-\frac{\partial\hat{\jmath}}{\partial\tau} &\text{(C.20)}\\
&+\phi\mathbb{E}_s^X\left\{\left[\Theta\left(V\left[\hat{f}\left(\cdot,s\right),X\left(s\right)\right]\right)+\theta\left(V\left[\hat{f}\left(\cdot,s\right),X\left(s\right)\right]\right)\int_0^T\hat{\mu}\left(m,s\right)\psi(m,s)dm\right]j\left(\tau,s\right)-\hat{\jmath}\left(\tau,s\right)\right\} &\text{(C.21)}\\
\hat{\jmath}\left(0,s\right)&=-U'\left(\hat{c}\left(s\right)\right).
\end{aligned}$$

Step 2.3. Gâteaux derivative with respect to the bond price. In the case of the Gâteaux derivatives with respect to the evolution of the price $\hat{\psi}$, $\frac{d}{d\alpha}\,\mathcal{L}\left[\hat{\imath},f,\hat{\psi}+\alpha h\right]\big|_{\alpha=0}$, we need to work first with the Lagragian before expectations have been computed. The reason is the following: only bonds that mature after default can be affected by the Governement's policies and hence the variations have to be zero for those bonds that mature before default, $\tilde{h}(\tau,t)=0$, if $\tau+t<t^o$. To incorporate this, we assume that admissible perturbations are of the form $\tilde{h}(\tau,t)=h(\tau,t)1_{\{\tau+t\geq t^o\}}$,

where $h(\tau,t)$ is unrestricted. The Gâteaux derivative is then

$$
\begin{aligned}
&\mathbb{E}^{t^o}\left[\int_0^{t^o} e^{-\rho s}U'\left(\hat{c}\left(s\right)\right)\left(\int_0^T \hat{\imath}\left(\tau,s\right)\frac{\partial q}{\partial\hat{\psi}}1_{\{\tau+s\geq t^o\}}h\left(\tau,s\right)d\tau\right)ds\right.\\
&+\int_0^{t^o}\int_0^T e^{-\rho s}\hat{\mu}\left(\tau,s\right)\left(-\hat{r}^*\left(s\right)1_{\{\tau+s\geq t^o\}}h(\tau,s)\right)d\tau ds\\
&-\int_0^T \hat{\mu}\left(\tau,0\right)1_{\{\tau\geq t^o\}}h\left(\tau,0\right)d\tau\\
&-\int_0^{t^o}\int_0^T e^{-\rho s}1_{\{\tau+s\geq t^o\}}h(\tau,s)\left(\frac{\partial\hat{\mu}}{\partial s}-\rho\hat{\mu}(\tau,s)\right)d\tau ds\\
&-\int_0^{t^o} e^{-\rho s}\left[1_{\{T+s\geq t^o\}}h\left(T,s\right)\hat{\psi}\left(T,s\right)-e^{-\rho s}1_{\{s\geq t^o\}}h\left(0,s\right)\hat{\psi}\left(0,s\right)\right]ds\\
&\left.+\int_0^{t^o}\int_0^T e^{-\rho s}1_{\{\tau+s\geq t^o\}}h\left(\tau,s\right)\frac{\partial\hat{\mu}}{\partial\tau}d\tau ds\right].
\end{aligned}
$$

Note that the perturbation is only around $\hat{\psi}\left(\tau,s\right)$ and not $\psi\left(\tau,s\right)$, the terminal price after default, which is given. Since at maturity, bonds have a value of 1, $h\left(0,s\right)=0$, because no perturbation can affect that price. If we compute the expectation with respect to the random arrival time, t^o, we get:

$$
\begin{aligned}
&\int_0^\infty e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)U'\left(\hat{c}\left(s\right)\right)\left[\int_0^T \hat{\imath}\left(\tau,s\right)\frac{\partial q}{\partial\hat{\psi}}h\left(\tau,s\right)d\tau\right]ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)\hat{\mu}\left(\tau,s\right)\left(-\hat{r}^*\left(s\right)h(\tau,s)\right)d\tau ds\\
&-\int_0^T\left(1-e^{-\phi\tau}\right)\mu\left(\tau,0\right)h\left(\tau,0\right)d\tau\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)h(\tau,s)\left(\frac{\partial\hat{\mu}}{\partial s}-\rho\mu(\tau,s)\right)d\tau ds\\
&-\int_0^\infty e^{-(\rho+\phi)s}\left(1-e^{-\phi T}\right)\mu\left(T,s\right)h\left(T,s\right)ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)\hat{\psi}\left(\tau,s\right)\frac{\partial\hat{\mu}}{\partial\tau}d\tau ds,
\end{aligned}
$$

where we use

$$\mathbb{E}^{t^o}\left[1_{\{\tau+s>t^o>s\}}\right]=e^{-\phi s}\left(1-e^{-\phi\tau}\right).$$

Again, as the Gâteaux derivative should be zero for any suitable $h(\tau,s)$, the optimality condition is

$$
\begin{aligned}
\left(\hat{r}^*\left(s\right)-\rho\right)\hat{\mu}(\tau,s) &= U'\left(\hat{c}\left(s\right)\right)\hat{\imath}\left(\tau,s\right)\frac{\partial q}{\partial\hat{\psi}}-\frac{\partial\hat{\mu}}{\partial s}+\frac{\partial\hat{\mu}}{\partial\tau},\\
\hat{\mu}\left(T,s\right) &= 0,\\
\hat{\mu}\left(\tau,0\right) &= 0.
\end{aligned}
$$

The solution to this PDE is

$$\hat{\mu}\left(\tau,s\right)=\int_{\max\{s+\tau-T,0\}}^s e^{-\int_z^s(\hat{r}^*(u)-\rho)du}U'\left(\hat{c}\left(z\right)\right)\hat{\imath}\left(\tau+s-z,z\right)\frac{\partial q}{\partial\hat{\psi}}\left(\tau+s-z,z\right)dz.$$

If we integrate the discount factor of the government with respect to time, we obtain the following identity:

$$\int_z^s \hat{r}(u)\,du = \int_z^s \rho du - \int_z^s \frac{U''(\hat{c}(u))}{U'(\hat{c}(u))}\hat{c}(u)\frac{\dot{\hat{c}}(u)}{\hat{c}(u)}du.$$

Therefore, we have that

$$\int_z^s \hat{r}(u)\,du - \int_z^s \rho du = -\,log\left(U'(\hat{c}(u))\right)\big|_z^s.$$

We obtain the following identity:

$$e^{\int_z^s \rho du} = e^{\int_z^s \hat{r}(u)du}\frac{U'(\hat{c}(s))}{U'(\hat{c}(z))}.$$

Thus, the PDE for $\hat{\mu}(\tau,s)$ can be written as:

$$\hat{\mu}(\tau,s) = U'(\hat{c}(s))\int_{\max\{s+\tau-T,0\}}^s e^{-\int_z^s(\hat{r}^*(u)-\hat{r}(u))du}\hat{\iota}(\tau+s-z,z)\frac{\partial q}{\partial\hat{\psi}}(\tau+s-z,z)\,dz.$$

Notice that

$$\begin{aligned}
\hat{\iota}\frac{\partial q}{\partial\hat{\psi}} &= \left(\hat{\iota}-\frac{\bar{\lambda}}{2}(\hat{\iota})^2\right)\\
&= \frac{1}{2\bar{\lambda}}\left(\frac{\hat{\psi}^2-\hat{v}^2}{\hat{\psi}^2}\right)\\
&= \frac{1}{2\bar{\lambda}}\left(1-\frac{\hat{v}^2}{\hat{\psi}^2}\right).
\end{aligned}$$

where the second line uses the optimal issuance rule. Hence, we can write the price multiplier as:

$$\hat{\mu}(\tau,s) = U'(\hat{c}(s))\int_{\max\{s+\tau-T,0\}}^s e^{-\int_z^s(\hat{r}^*(u)-\hat{r}(u))du}\frac{1}{2\bar{\lambda}}\left(1-\left(\frac{\hat{v}(\tau+s-z,z)}{\hat{\psi}(\tau+s-z,z)}\right)^2\right)dz.$$

We employ this solution in the main text.

Step 3: From Lagrange multipliers to valuations. We now employ the definitions of $\hat{v}(\tau,s) = -\hat{j}(\tau,s)/U'(\hat{c}(s))$ and $v(\tau,s) = -j(\tau,s)/U'(c(s))$, we can express equations (C.20)-(C.21) as

$$\begin{aligned}
\hat{r}(s)\hat{v}(\tau,s) &= \delta+\frac{\partial\hat{v}}{\partial s}-\frac{\partial\hat{v}}{\partial\tau}\\
&\quad+\phi\mathbb{E}_s^X\left\{\left[\Theta(V(s))+\theta(V(s))\int_0^T\hat{\mu}(m,s)\,\psi(m,s)dm\right]\frac{U'(c(s))}{U'(\hat{c}(s))}v(\tau,s)-\hat{v}(\tau,s)\right\},\\
\hat{v}(0,s) &= 1,
\end{aligned}$$

where we use the notation $\Theta(V(s)) \equiv \Theta(V[f(\cdot,s),X(s)])$ and $\theta(V(s)) \equiv \theta(V[f(\cdot,s),X(s)])$. Therefore valuations can be expressed as

$$\begin{aligned}
\hat{v}(\tau,t) &= e^{-\int_t^{t+\tau}(\hat{r}(s)+\phi)ds}\\
&\quad+\phi\int_t^{t+\tau}e^{-\int_t^s(\hat{r}(u)+\phi)du}\left(\delta+\mathbb{E}_s^X\left[(\Theta(V(t+s))+\Omega(t+s))\frac{U'(c(t+s))}{U'(\hat{c}(t+s))}v(\tau-s,t+s)\right]\right)ds,
\end{aligned}$$

where

$$\Omega(t) = \theta(V(t))\int_0^T\hat{\mu}(m,t)\,\psi(m,t)dm.$$

□

C.8 The case of default without liquidity costs: $\bar{\lambda} = 0$

We show here that the maturity structure is indetermined in the case without liquidity costs and a finite support of G. In proposition 9 below we show how, if distribution $\hat{f}^*$ is a solutions of Problem (4.3), then another distribution $\hat{f}'$ is also a solution provided that

$$\int_0^T \left(\psi\left(\tau, t, X(t)\right) - \hat{\psi}\left(\tau, t\right)\right) \left(\hat{f}^*(\tau, t) - \hat{f}'(\tau, t)\right) d\tau = 0, \tag{C.22}$$

$$\int_0^T \left(\mathbb{E}_{X_t}\left[\Theta\left(V\left[\hat{f}^*\left(\cdot, t\right), X(t)\right]\right) \psi\left(\tau, t, X(t)\right)\right] - \hat{\psi}\left(\tau, t\right)\right) \left(\hat{f}^*(\tau, t) - \hat{f}'(\tau, t)\right) d\tau = 0. \tag{C.23}$$

Consider first the case of an income shock. Here $X(t)$ does not jump after the shock arrives. If the government decides to default then the maturity profile at the moment of defaut is irrelevant. If the government decides instead to repay, the post-shock yield curve will be $\psi\left(\tau, t^o\right)$, which differs from $\hat{\psi}\left(\tau, t^o\right)$ as the post-shock default premium is zero. The maturity structure is indeterminate because conditions (C.22) and (C.23) are two integral equation with a continuum of unknowns, $\hat{f}'\left(\tau, t^o\right)$.

Consider next the case of an interest rate shock, in which $X(t)$ jumps with the option to default. Condition (C.22) is a system of integral equations, indexed by $X(t^o)$, where $\hat{f}'\left(\cdot, t^o\right)$ is the unknown. Provided that $X(t^o)$ may take N possible values, then we have at most N equations that need to be satisfied by the debt distribution. In addition we have equation (C.23) and the condition that the market debt should coincide. Notice that the number can be less than $N + 2$ as in some states the government may default and then condition C.22 is trivially satisfied for any debt profile that replicates the total debt at market prices before the shock arrival. In any case, the maturity structure is indeterminate.

The indeterminacy of the debt distribution in our model complements previous results in the literature. In particular, Aguiar et al. (Forthcoming) study a model of sovereign default similar to the one presented with the key difference that in their model the government cannot commit to future debt issuances whereas in our paper it ca, conditional on repayment. Aguiar et al. (Forthcoming) find how in that case the government only operates in the short end of the curve, making payments and retiring long-term bonds as they mature but never actively issuing or buying back such bonds. This is because short term bonds cannot be diluted. The authors also conjecture that the maturity structure would be indeterminate if the government had full commitment over its issuance path. This is precisely the case we study here, confirming their conjecture.

Proposition 9. *Let* $\{\hat{\iota}^*(\tau,t), \hat{f}^*(\tau,t), \hat{c}^*(t)\}_{t\in[0,t^o]}$ *and* $\{\iota^*(\tau,t), f^*(\tau,t), c^*(t)\}_{t\in(t^o,\infty)}$ *be the solution of Problem (4.3) when* $\lambda\left(\iota, \tau, t\right) = 0$. *Let* $\{\hat{\iota}'(\tau,t), \hat{f}'(\tau,t), \hat{c}'(t)\}_{t\in[0,t^o]}$ *and* $\{\iota^*(\tau,t), f^*(\tau,t), c^*(t)\}_{t\in(t^o,\infty)}$ *be such that, for every* $t \le t^o$ *and every value of* $X(t)$,

$$\hat{B}^*(t) = \hat{B}'(t) \tag{C.24}$$

$$\int_0^T \left(\psi\left(\tau, t, X(t)\right) - \hat{\psi}\left(\tau, t\right)\right) \left(\hat{f}^*(\tau, t) - \hat{f}'(\tau, t)\right) d\tau = 0, \tag{C.25}$$

$$\int_0^T \left(\mathbb{E}_{X_t}\left[\Theta\left(V\left[\hat{f}^*\left(\cdot, t\right), X(t)\right]\right) \psi\left(\tau, t, X(t)\right)\right] - \hat{\psi}\left(\tau, t\right)\right) \left(\hat{f}^*(\tau, t) - \hat{f}'(\tau, t)\right) d\tau = 0, \tag{C.26}$$

and $B^*(t) = B'(t)$ *for every* $t > t^o$. *Then,* $\hat{c}'(t) = \hat{c}^*(t)$ *and* $c'(t) = c^*(t)$. *Thus,* $\{\hat{\iota}'(\tau,t), \hat{f}'(\tau,t), \hat{c}'(t)\}_{t\in[0,t^o]}$ *and* $\{\iota^*(\tau,t), f^*(\tau,t), c^*(t)\}_{t\in(t^o,\infty)}$ *is also optimal.*

Proof. Step 0. Default values. The value functional of a policy given an initial debt $f(\cdot,0)$ is given by:

$$\hat{V}[f(\cdot,0)] = \mathbb{E}_0\left[\int_0^{t^o} e^{-\rho t}U(\hat{c}(t))\,dt + \mathbb{E}_{V^D,X(t^o)}\left[e^{-\rho t^o}V^O\left[V^D(t^o), f(\cdot,t^o)\right]\right]\right]$$

where the post-default value $V^O\left[V^D(t^o), f(\cdot,t^o), X(t^o)\right] \equiv \max\left\{V^D(t^o), V[f(\cdot,t^o), X(t^o)]\right\}$ and $V[f(\cdot,t^o), X(t^o)]$ is the value of the perfect-foresight solution. Note that, from the solution of the problem with perfect foresight, the value function only depends on the market value of total debt, $V[f(\cdot,t^o), X(t^o)] = V(B(t^o, X(t^o)), X(t^o))$, where $B(t^o, X(t^o))$ is defined as the market value of debt

$$B(t^o, X(t^o)) \equiv \int_0^T \psi(\tau, t^o, X(t^o))\, f(\tau, t^o) d\tau.$$

Therefore, the post-default value

$$V^O\left[V^D(t^o), f(\cdot,t^o), X(t^o)\right] = V^O\left(V^D(t^o), B(t^o, X(t^o)), X(t^o)\right),$$

also only depends on the aggregate market value of total debt, $B(t^o, X(t^o))$. Because $B'(t^o, X(t^o)) = B^*(t^o, X(t^o))$ for every realization of $X(t^o)$ the default decision depends only on the market value of debt when the country receives the opportunity to default and not on the debt-maturity profile. Thus, continuation values are equal and it is enough to show that $\hat{c}^*(t) = \hat{c}'(t)$ for $t \leq t^o$ to prove that the two policies yield the same utility.

Step 1. Pre-shock prices are equal. Pre-shock prices solve

$$\begin{aligned}
\hat{r}^*(t)\,\hat{\psi}(\tau,t) &= \delta + \frac{\partial\hat{\psi}}{\partial t} - \frac{\partial\hat{\psi}}{\partial\tau} + \phi\mathbb{E}_t^X\left[\Theta\left(V[f(\cdot,t), X(t)]\right)\psi(\tau,t,X(t)) - \hat{\psi}(\tau,t)\right], \text{ if } t < t^o \\
\hat{\psi}(\tau, t^o) &= \mathbb{E}_{t^o}^X\left\{\Theta\left(V\left[\hat{f}(\cdot,t^o), X(t^o)\right]\right)\psi(\tau,t^o)\right\} \\
\hat{\psi}(0,t) &= 1.
\end{aligned}$$

It holds that

$$\Theta\left(V[f^*(\cdot,t^o), X(t^o)]\right) = \Theta\left(V(B^*(t^o, X(t^o)), X(t^o))\right) = \Theta\left(V(B'(t^o, X(t^o)), X(t^o))\right) = \Theta\left(V[f'(\cdot,t^o), X(t^o)]\right).$$

This is a consequence of the fact that $V(B^*(t^o, X(t^o)), X(t^o)) = V(B'(t^o, X(t^o)), X(t^o))$. Thus, pre-shock prices are equal for both policies.

Step 2. Law of of motion of debt before the shock arrival. By definition $\hat{B}(t) = \int_0^T \hat{\psi}(\tau,t)\hat{f}(\tau,t)d\tau$. The dynamics of $\hat{B}(t)$ for $t < t^o$ are:

$$d\hat{B}(t) = \left(\int_0^T \left(\hat{\psi}_t(\tau,t)\hat{f}(\tau,t) + \hat{\psi}(\tau,t)\hat{f}_t(\tau,t)\right)d\tau\right)dt,$$

which, with similar derivations as in 7, yields to

$$d\hat{B}(t) = \left(\hat{c}(t) - y(t) + \hat{r}^*(t)\hat{B}(t) + \phi\int_0^T \left(\mathbb{E}_{X_t}\left[\Theta\left(V(B(t,X(t)), X(t))\right)\psi(\tau,t,X(t))\right] - \hat{\psi}(\tau,t)\right)\hat{f}(\tau,t)d\tau\right)dt.$$

Step 3. The expected jump. Note that (C.26) for all X_t implies that:

$$\phi\int_0^T \left(\mathbb{E}_{X_t}\left[\Theta\left(V(B^*(t,X(t)), X(t))\right)\psi(\tau,t,X(t))\right] - \hat{\psi}(\tau,t)\right)\left(\hat{f}(\tau,t) - \hat{f}'(\tau,t)\right)d\tau = 0, \tag{C.27}$$

Combining this equation with the law of motion of debt we get that before the shock arrival, $t < t^o$,

$$d\hat{B}^*(t) = d\hat{B}'(t). \tag{C.28}$$

Step 4. The actual jump. Condition (C.25) guarantees that the jump is the same for any $X(t)$ if the country does not default. If it defaults, condition (C.25) is trivially satisfied as $\hat{B}^*(t) = \hat{B}'(t)$ and the jump is also the same as market debt is then zero. Hence $\hat{B}^*(t^o) = \hat{B}'(t^o)$. Finally, taking all these results together we conclude that: $\hat{c}^*(t) = \hat{c}'(t)$ for all $t \leq t^o$. As the policy $\{\hat{\iota}'(\tau,t), \hat{f}'(\tau,t), \hat{c}'(t)\}_{t\in[0,t^o]}$ and $\{\iota^*(\tau,t), f^*(\tau,t), c^*(t)\}_{t\in(t^o,\infty)}$ achieves the same consumption path as the optimal, it is thus optimal. $\square$

D Calibration notes

In this Appendix, we describe the sources of the data and the calibration procedure for the parameter values and shocks used in the numerical exercises in subsection 2.6, subsection 3.3, and subsection 4.3.

Income Process: ρ_y

We obtain the series for the Spanish Gross Domestic Product for the period Q1:1995 to Q2:2018 from FRED economic data `https://fred.stlouisfed.org/series/CLVMNACSCAB1GQES`. We estimate an AR(1) process for the detrended seasonally adjusted output. The detrending uses a Hodrick-Prescott filter with a parameter of 1600. The estimated model is $\log y_t = \rho_y \log y_{t-1} + \sigma_y \epsilon^y_{t-1}$. The estimated persistence of quarterly income (ρ_y) is 0.95 with a standard deviation (σ_y) of 0.375. This corresponds to a value of $\alpha_y = (1-\rho_y)/\,(3\times\Delta t) = 0.2$, where we fix $\Delta t = 1/12$ for all our numerical exercises. The details of the numerical procedure to solve the model are described in Appendix F.

Interest Rates Process: ρ_r

We obtain from the Bundesbank monthly data for the 1 month Euribor nominal rate. The period is Q1 1999 to Q2 2018. The source of the data is: `https://www.bundesbank.de/action/en/744770/bbkstatisticsearch?query=euribor`. We then obtain the annualized average (geometric mean) quarterly rate, i^q_t, as $\left[\Pi^3_{m=1}(1+i^m_t)\right]^{\frac{1}{3}} = (1+i^q_t)$. Using the quarterly inflation rate for the Eurozone (overall index not seasonally adjusted), obtained from the ECB Statistical Data Warehouse, the source of the data is `http://sdw.ecb.europa.eu/`, we compute the annualized real rate at a quarterly frequency as $\frac{1+i^q_t}{1+\pi^q_t} = 1 + r^q_t$. We then fit an AR(1) process for the level of the real interest rate $r_t = \mu_r + \rho_r r_{t-1} + \sigma_r \epsilon^r_{t-1}$. The estimated persistence of the quarterly real rate (ρ_r) is 0.95, which translates into $\alpha_r = (1-\rho_r)/\,(3\times\Delta t) = 0.2$, where $\Delta t = 1/12$. The standard deviation (σ_r) is equal to 0.410.

Dealers Cost of Capital: η

We approximate the cost of capital of dealers, η, as follows. For each one of the five largest US banks by Assets, we obtain the current (as of August 2018) credit rating from Fitch, Standard and Poor's and Moody's. At the same time we obtain AA, A, BBB daily option-adjusted spreads (OAS) for US corporate bonds from FRED Economic Data for the period January 1st of 1997 to August 27th of 2018. The source of the data is: `https://fred.stlouisfed.org/search?st=ICE+BofAML+US+Corporate+AA+Option-Adjusted+Spread`. Options adjusted spreads measure the spread of a family of US Corporate issuers of the same credit rating adjusting for

Parameter	Description	Value
γ	Sovereign's risk aversion	2
δ	Coupon rate	0.04
α_y	Persistence of output	0.2
α_r	Persistence of short rate	0.2
μ	Arrival rate	0.0011
η	Cost of Capital for Intermediaries (pct/py)	1.49
$\bar{\lambda}$	Implied liquidity cost	7.08
ρ	Discount Factor	0.0416
ϕ	Poisson Int. of a Large Shock (pct/py)	2.00
Δy	Drop in output (pct)	5.00
Δr	Increase in rates (pct)	1.00
ς	Logistic p.d.f. scale parameter	100

Table 2: Summary Baseline Calibration

any embedded option.[30] We denote $OAS_t(\text{Rating})$ as a function that maps a rating and a moment in time to the option-adjusted spread. For each bank i we fix the current credit rating, $\text{Rating}_{i,t}$, for the entire sample of daily observations at the final value, $\text{Rating}_{i,T}$. Then, we obtain a time series for the spread of each bank as $\eta_{i,t} = OAS_t(\text{Rating}_{i,T})$ for $t = 1....T$. We weight the spreads of each bank by their relative assets (as of 2018). Finally, we obtain an estimated value of the spread of intermediaries in our model as $\hat{\eta} = \frac{\sum_{t=1}^{T}\sum_{i=1}^{5}\omega_{i,T}\eta_{i,t}}{T}$, equal to 149 basis points.

Spanish Debt Profile: Figure 2.1

For the maturity debt profile of Spain featured in figure 2.1, we use data from the Spanish Treasury. The data can be accessed at `www.tesoro.es/en/deuda-publica/estadísticas-mensuales` and corresponds to the debt profile as of July the 31st of 2018. We use the total debt.

Large Shock Intensity: ϕ

We calibrate the intensity of the shock based on the data from Barro and Ursua (2010). This data-set is obtained from `https://scholar.harvard.edu/barro/publications/barro-ursua-macroeconomic-data`. Out of 1600 year-country observations for OECD countries, 34 of them correspond to an output drop of more than 5 percent. This amounts to an estimated frequency equal to 2.13 percent per year; one large shock every 50 years. We calibrate the arrival of shocks in the risky steady state, ϕ, to a value equal to 2.00 percent per year.

Large Shock Size: $\Delta y(0), \Delta r(0)$

We fix 5 percent as the size of the shock to output, and we denote it as $\Delta y(0)$. For interest rates, the shock to the short rate is the one that implies the same drop in consumption than with the shock to output, given the persistence of rates (as well as the other parameters). This is an increase in rates from 4 to 5 percent, and we denote it as $\Delta r(0)$.

[30] Bank of America Merrill Lynch computes the index. More precisely, the option-adjusted spread for bonds is "the number of basis points that the fair value government spot curve is shifted to match the present value of discounted cash flows to the bond price. For securities with embedded options, such as call, sink or put features, a log-normal short interest rate model is used to evaluate the present value of the securities potential cash flows." Thus, the index fits a term structure model for different securities, to separate the value of the security from any embedded option by subtracting or adding the value of the option.

Scale Logistic Distribution: ς

To calibrate ς we match the unconditional default probability of Spain during the period 1877-1982. We proceed as follows. According to Barro and Ursua (2010), for the period 1945 to 2009 the most significant year-to-year drop in income for Spain was 4.8 percent. Thus, we will fix the size of the shock output to 5.0 percent and the intensity to $\phi = 0.02$, as in section 3. The preference shocks are distributed according to a logistic distribution with a probability density function given by:

$$f(\varepsilon) = \frac{\varsigma e^{-\varsigma\varepsilon}}{\left(1 + e^{-\varsigma\varepsilon}\right)^2}.$$

We set the scale parameter, ς, equal to 100. As we mentioned in the main text, for our calibration, this value of the parameter produces a default in 32 percent of the events when an extreme shock hits Spanish output. Given the intensity of the extreme shock, $\phi = 0.02$, this implies an unconditional default probability equal to 0.6 percent per year, roughly a default every 157 years. This is in line with the findings of Reinhart and Rogoff (2009) in which Spain experienced one default during the period 1877-1982.

E Some stylized facts bond issuances in Spain

We report data on the monthly gross issuances of bonds by the Spanish Government. The period covers from January 2000 to December 2018. The source of the data is the Spanish Treasury. The data can be accessed at `www.tesoro.es/en/deuda-publica/historico-de-estadísticas/subastas-2001-2014`. In this period the Treasury issued debt in bills (zero-coupon bonds) and regular bonds of different maturities. Bills are issued at 3, 6, 12 and 18-month maturities, and bonds are issued at 3, 5, 10, 15 and 30-year maturities.[31] Not all maturities are active every month. For example, in the period 2014-2018 the 18-month bill and the 30-year bond were not issued. We construct a panel of annual issuances over GDP by accumulating individual gross issuances, and dividing them by the Spanish GDP in current euros. The source for GDP data is FRED economic data `https://fred.stlouisfed.org/series/CLVMNACSCAB1GQES`.

Figure E.1 reports the yearly average of issuances at each maturity over GDP. A pattern can be observed from the data. (i) Maturity is increasing for all bonds up to one year. (ii) For bonds from 3 years up to 10 years, maturity is also increasing, even if the absolute levels are below the amount of 1-year bonds. (iii) The issuance of 15 and 30-year bonds is roughly similar and relatively small. The figure also displays issuances weighted by the total issuance over GDP of the corresponding year. The pattern is similar independently of the weighting.

Another remarkable feature is that this pattern has been relatively stable over time. Figure E.2 presents the yearly averages for each vintage. Notwithstanding the changes in the total level of issuances, there is a remarkable stability in the issuance pattern that we highlight in the figure E.1. For instance, the bottom right panel displays the issuances in the period 2014-2018 of low interest rates and quantitative easing. Notice how, even if issuances have shifted towards long maturities, the issuance pattern remains roughly similar to the average.

With a liquidity coefficient $\bar{\lambda}$ constant across maturities our model cannot replicate this pattern of maturity increasing within groups of bonds and discrete issuances. However, a natural way to fit the data, without over-parameterizing the model, would be to allow for discrete issuances, as outlined in section 5, and to consider heterogeneity in the liquidity coefficient among different groups of bonds. For example, we could let order flows (captured by μ) to differ for bonds with maturities below 1 year, between 1 and 10 years, and above 10 years. This variation of the model can be rationalized if bonds trade in markets that differ in their liquidity or ability to be

[31] The Treasury has occasionally issued at other maturities, including recent issuances of 50-year bonds, but we do not consider them in the analysis.

pledged as collateral (see Krishnamurthy and Vissing-Jorgensen, 2012), or if they are catered to customers with different investment horizons (see, for example, Vayanos and Vila, 2009).

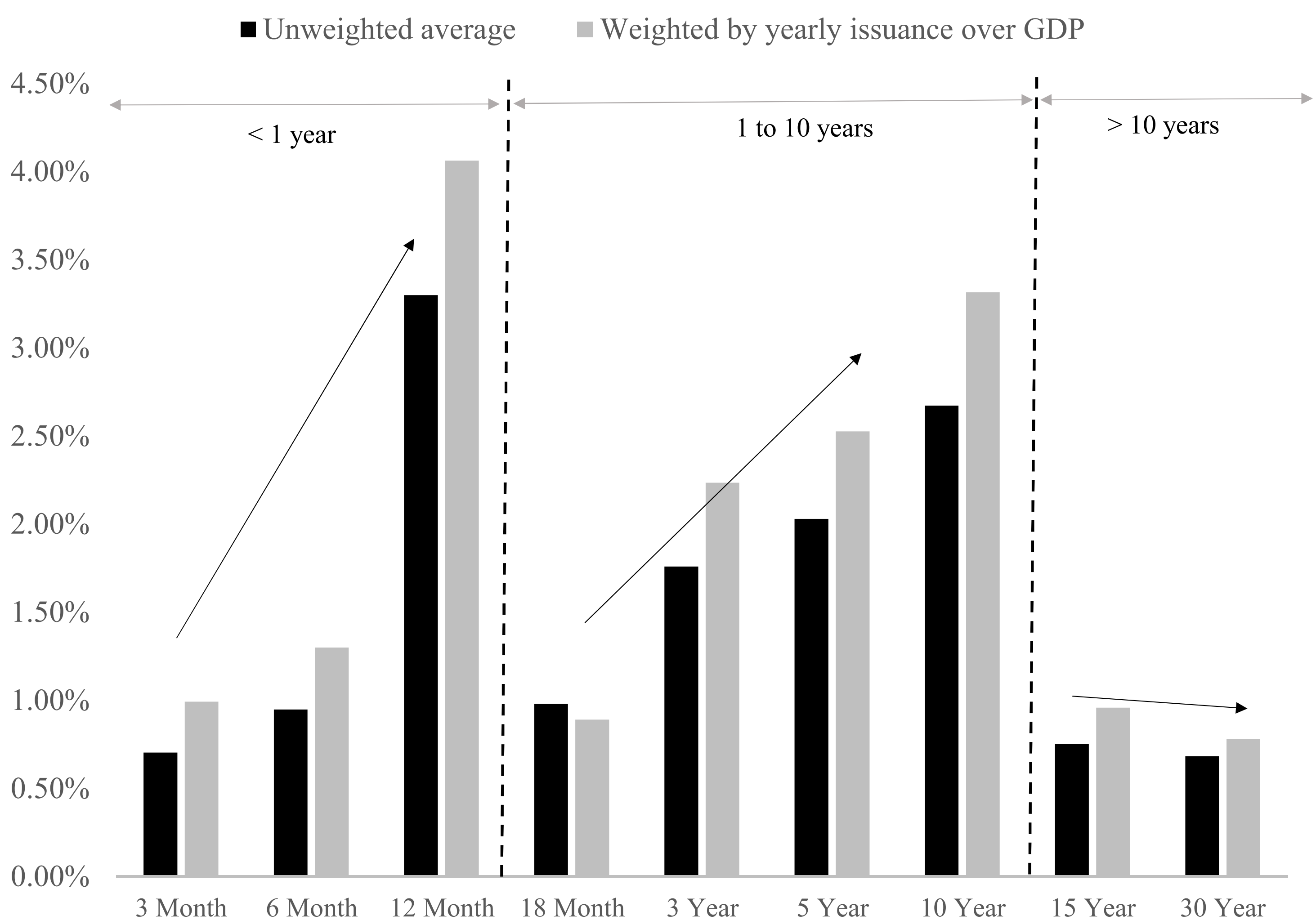

Figure E.1: Issuance by maturity as a percentage GDP. Sample average for Spain 2000 - 2018

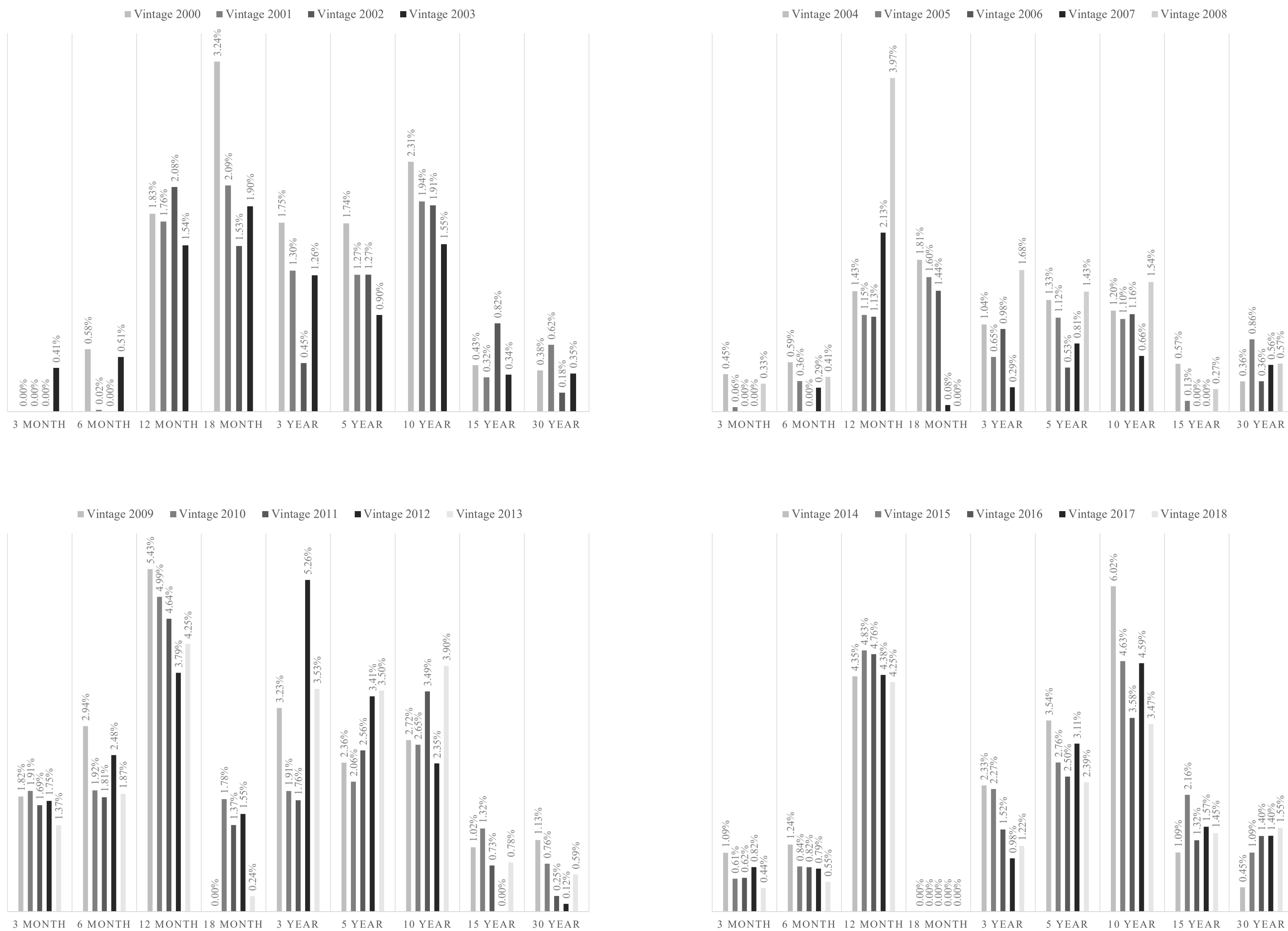

Figure E.2: Issuance by maturity as a percentage of GDP. Vintages 2000 - 2018

F Computational method

We describe the numerical algorithm used to jointly solve for the equilibrium domestic valuation, $v(\tau, t)$, bond price, $q(t, \tau, \iota)$, consumption $c(t)$, issuance $\iota(\tau, t)$ and density $f(\tau, t)$. The initial distribution is $f(\tau, 0) = f_0(\tau)$. The algorithm proceeds in 3 steps. We describe each step in turn.

Step 1: Solution to the domestic value The steady state equation (2.10) is solved using an *upwind finite difference* scheme similar to Achdou et al. (2017). We approximate the valuation $v_{ss}(\tau)$ on a finite grid with step $\Delta\tau : \tau \in \{\tau_1, ..., \tau_I\}$, where $\tau_i = \tau_{i-1} + \Delta\tau = \tau_1 + (i-1)\Delta\tau$ for $2 \leq i \leq I$. The bounds are $\tau_1 = \Delta\tau$ and $\tau_I = T$, such that $\Delta\tau = T/I$. We use the notation $v_i := v_{ss}(\tau_i)$, and similarly for the issuance $\iota_{i,}$. Notice first that the domestic valuation equation involves first derivatives of the valuations. At each point of the grid, the first derivative can be approximated with a forward or a backward approximation. In an upwind scheme, the choice of forward or backward derivative depends on the sign of the *drift function* for the state variable. As in our case, the drift is always negative, we employ a backward approximation in state:

$$\frac{\partial v(\tau_i)}{\partial \tau} \approx \frac{v_i - v_{i-1}}{\Delta\tau}. \tag{F.1}$$

The equation is approximated by the following upwind scheme,

$$\rho v_i = \delta + \frac{v_{i-1}}{\Delta\tau} - \frac{v_i}{\Delta\tau},$$

with terminal condition $v_0 = v(0) = 1$. This can be written in matrix notation as

$$\rho \mathbf{v} = \mathbf{u} + \mathbf{A}\mathbf{v},$$

where

$$\mathbf{A} = \frac{1}{\Delta\tau}\begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{I-1} \\ v_I \end{bmatrix}, \mathbf{u} = \begin{bmatrix} \delta - 1/\Delta\tau \\ \delta \\ \delta \\ \vdots \\ \delta \\ \delta \end{bmatrix}. \tag{F.2}$$

The solution is given by

$$\mathbf{v} = (\rho \mathbf{I} - \mathbf{A})^{-1}\mathbf{u}. \tag{F.3}$$

Most computer software packages, such as Matlab, include efficient routines to handle sparse matrices such as $\mathbf{A}$.

To analyze the transitional dynamics, define $t^{\max}$ as the time interval considered, which should be large enough to ensure a converge to the stationary distribution and time is discretized as $t_n = t_{n-1} + \Delta t$, in intervals of length

$$\Delta t = \frac{t^{\max}}{N-1},$$

where N is a constant. We use now the notation $v_i^n := v(\tau_i, t_n)$. The valuation at $t^{\max}$ is the stationary solution computed in (F.3) that we denote as $\mathbf{v}^N$. We choose a forward approximation in time. The dynamic value equation (2.10) can thus be expressed

$$r^n \mathbf{v}^n = \mathbf{u} + \mathbf{A}\mathbf{v}^n + \frac{\left(\mathbf{v}^{n+1} - \mathbf{v}^n\right)}{\Delta t},$$

where $r^n := r(t_n)$. By defining $\mathbf{B}^n = \left(\frac{1}{\Delta t} + r^n\right)\mathbf{I} - \mathbf{A}$ and $\mathbf{d}^{n+1} = \mathbf{u} + \frac{\mathbf{v}^{n+1}}{\Delta t}$, we have

$$\mathbf{v}^n = (\mathbf{B}^n)^{-1}\,\mathbf{d}^{n+1}, \tag{F.4}$$

which can be solved backwards from $n = N-1$ until $n = 1$.

The optimal issuance is given by

$$\iota_i^n = \frac{1}{\bar{\lambda}}\frac{\left(\psi_i^n - v_i^n\right)}{\psi_i^n},$$

where ψ_i^n is computed in an analogous form to v_i^n.

Step 2: Solution to the Kolmogorov Forward equation Analogously, the KFE equation (2.1) can be approximated as

$$\frac{f_i^n - f_i^{n-1}}{\Delta t} = \iota_i^n + \frac{f_{i+1}^n - f_i^n}{\Delta \tau},$$

where we have employed the notation $f_i^n := f(\tau_i, t_n)$. This can be written in matrix notation as:

$$\frac{\mathbf{f}^n - \mathbf{f}^{n-1}}{\Delta t} = \mathbf{i}^n + \mathbf{A}^{\mathbf{T}}\mathbf{f}^n, \tag{F.5}$$

where $\mathbf{A}^{\mathbf{T}}$ is the transpose of $\mathbf{A}$ and

$$\mathbf{f}^n = \begin{bmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_{I-1}^n \\ f_I^n \end{bmatrix}, \mathbf{i}^n = \begin{bmatrix} \iota_1^n \\ \iota_2^n \\ \vdots \\ \iota_{I-1}^n \\ \iota_I^n \end{bmatrix}.$$

Given $\mathbf{f}_0$, the discretized approximation to the initial distribution $f_0(\tau)$, we can solve the KF equation forward as

$$\mathbf{f}_n = \left(\mathbf{I} - \Delta t\mathbf{A}^{\mathbf{T}}\right)^{-1}\left(\mathbf{i}^n\Delta t + \mathbf{f}_{n-1}\right), \;\; n = 1, .., N. \tag{F.6}$$

Step 3: Computation of consumption The discretized budget constraint (2.2) can be expressed as

$$c^n = \bar{y}^n - f_1^{n-1} + \sum_{i=1}^{I}\left[\left(1_n - \frac{1}{2}\bar{\lambda}\iota_i^n\right)\iota_i^n\psi_i^n - \delta f_i^n\right]\Delta\tau, \;\; n = 1, .., N.$$

Compute

$$r^n = \rho + \frac{\sigma}{c^n}\frac{c^{n+1} - c^n}{\Delta t}, n = 1, .., N-1.$$

Complete algorithm The algorithm proceeds as follows. First guess an initial path for consumption, for example $c^n = \bar{y}^n$, for $n = 1, .., N$. Set $k = 1$;

Step 1: Issuances. Given c_{k-1} solve step 1 and obtain ι.

Step 2: **KF.** Given ι solve the KF equation with initial distribution f_0 and obtain the distribution f.

Step 3: **Consumption**. Given ι and f compute consumption c. If $\|c - c_{k-1}\| = \sum_{n=1}^{N}\left|c^n - c_{k-1}^n\right| < \varepsilon$ then stop. Otherwise compute

$$c_k = \omega c + (1-\omega)\,c_{k-1}, \; \lambda \in (0, 1),$$

set $k := k+1$ and return to step 1.

G Solutions without liquidity costs: risk and default

G.1 Risk without liquidity costs

We now consider the case without liquidity costs, $\bar{\lambda} = 0$. With positive liquidity costs, adjustments in portfolios are costly. By studying the problem at the limit where liquidity costs are zero, we can understand the extent to which the government can obtain insurance given the set of bonds it has available. Thus, it clarifies the extent to which liquidity costs limit insurance.

Toward that goal, we note that the necessary conditions of the problem are the same with and without liquidity costs, including the issuance rule. If the issuance rule holds, issuances are bounded if and only if valuations and prices are equal, $v = \psi$ and $\hat{v} = \hat{\psi}$. If we substitute $v = \psi$ and $\hat{v} = \hat{\psi}$ in the PDE for valuations, equation (3.3), and subtract the bond PDE from both sides, equation (3.1), we obtain a premium condition that must hold for all bonds:[32]

$$\hat{r}(t) - \phi \mathbb{E}_t^X \left[\frac{U'(c(t))}{U'(\hat{c}(t))} \cdot \frac{\psi(\tau,t)}{\hat{\psi}(\tau,t)} \right] = \hat{r}^*(t) - \phi \mathbb{E}_t^X \left[\frac{\psi(\tau,t)}{\hat{\psi}(\tau,t)} \right]. \tag{G.1}$$

The analysis of the different solutions of equation (G.1) provides useful information about the role of hedging and self-insurance. We analyze each case in turn.

Perfect hedging: replicating the complete-markets allocation. Equation (G.1) replicates the complete-markets allocation when consumption follows a continuous path, i.e., when $\hat{c}(t^o) = c(t^o; X(t^o))$ for any realization of the shock. This is because international investors are risk-neutral and the government is risk averse. If consumption does not jump, condition (G.1) implies $\hat{r} = \hat{r}^*$. In a complete markets economy consumption growth satisfies $\frac{\dot{c}(t)}{c(t)} = \frac{r^*(t)-\rho}{\sigma}$, the same rule that it follows in a deterministic problem. Naturally, there is no RSS with positive consumption if $r^*(t) < \rho$, but consumption converges asymptotically toward zero.

Consumption does not jump when it is possible to form a perfect hedge, a debt profile that generates a capital gain that exactly offsets the shock. Any shock changes the net-present value of income. Given the path of rates, the optimal consumption rule and the initial post-shock consumption produce a net-present value of consumption. A perfect hedge thus produces the capital gains such the net present value of consumption minus income at the time of the shock (denoted by $\Delta B(t^o, X(t^o))$) is covered to the point where pre- and post-shock consumption are equal: $c(t^o) = \hat{c}(t^o; X(t^o))$. This must be true for any shock, $X(t^o)$. In the context of the model, a perfect hedge exists if the debt distribution satisfies at all times t

$$\Delta B(t, X(t)) = -\int_0^T \left(\psi(\tau, t; X(t)) - \hat{\psi}(\tau,t)\right) \hat{f}(\tau,t)\, d\tau, \tag{G.2}$$

for any possible realization of $X(t)$. This family of equations is a generalization of the discrete-shock and discrete-bonds matrix conditions that guarantee market completion in Duffie and Huang (1985), Angeletos (2002) or Buera and Nicolini (2004).[33]

In our model, perfect hedging is available in the case of an interest rate shock taking N possible values. Then, there is continuum of solutions that satisfy equation (G.2). In this case we can use a range of maturities $[0, T]$ that is as short as we want to hedge. The shorter the range, the more extreme the positions we obtain. A second observation has to do with the direction of hedges. Consider the case of a single jump in interest rates ($N = 1$). To

[32] This equation is recovered also by solving the problem with $\lambda = 0$ directly. The proof is available upon request.

[33] In the case of discrete shocks and discrete bonds, the existence of complete-markets solution requires the presence of at least $N + 1$ bonds for N shocks. In the case of a continuum of shocks, the condition requires the invertibility of a linear operator. Proving conditions on G that guarantee that family of solutions exceeds the scope of the paper.In this case, equation (G.2) is just a system of N linear integral equations, for every t, known as Fredholm equations of the first kind.

offset the reduction in the net-present value of income, the debt profile must generate an increase in wealth. This requires an increase in short-term assets and long-term liabilities.

No hedging: only self insurance. The opposite to the complete-markets outcome is the case of income shocks. In this case bond prices do not change, $\psi = \hat{\psi} = 1$. Therefore, it is not possible to generate capital gains with a debt profile. Instead, the only solution to (G.1) is:

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \frac{\hat{r}^*(t) + \phi\left(\mathbb{E}_t^X\left[\frac{U'(c(t))}{U'(\hat{c}(t))}\right] - 1\right) - \rho}{\sigma}.$$

This is a situation in which no hedging is available, because the asset space does not allow any form of external insurance. Instead, the government must self-insure. Self insurance is captured by the ratio of marginal utilities which effectively lowers $\hat{r}(t)$. To solve for consumption, this extreme case coincides with a single-bond economy without interest-rate risk. The jump in consumption is given by the jump in the net present value of income. The solution to $c(t)$ in this case is known and can be found, for example in Wang et al. (2016). The ratio of marginal utilities in the solution increases as the level of assets falls. This means that, provided there is a sufficiently low level of debt, the economy reaches a RSS with positive consumption. The convergence in consumption is a manifestation of self-insurance.

General case. The general case with both income and interest rates shocks described by equation (G.1) features an intermediate point between the two extreme cases described above as both a partial hedging and self-insurance emerge.[34] Furthermore, as long as the support of the shocks has cardinality N the debt profile is indeterminate, as only N points of the debt distribution are pinned down.[35]

G.2 Default without liquidity costs

We return to the case without liquidity costs, $\bar{\lambda} = 0$, but allow for default. Without liquidity costs, we have again that a solution neccesarily features equality between valuations and prices, $v = \psi$ and $\hat{v} = \hat{\psi}$. As a result, the condition that characterizes the solution without default, (G.1), is modified to:

$$\hat{r}(t) - \phi\mathbb{E}_t^X\left[(\Theta(V(t)) + \Omega(t))\frac{\psi(\tau,t)}{\hat{\psi}(\tau,t)}\cdot\frac{U'(c(t))}{U'(\hat{c}(t))}\right] = \hat{r}^*(t) - \phi\mathbb{E}_t^X\left[\Theta(V(t))\frac{\psi(\tau,t)}{\hat{\psi}(\tau,t)}\right]. \tag{G.3}$$

As in the case without default, we can explain how condition (G.3) characterizes the solution depending on the set of bonds and shocks.

On the impossibility of perfect hedging. The presence of default interrupts the ability to share risk. Efficient risk sharing requires a continuous consumption path along non-default states. To see how default interrupts risk-sharing, consider the case where interest-rate shocks allow complete asset spanning. Assume that $\hat{c}(t) = c(t)$ holds in non-default states, as in the version without default. In this case, condition (G.3) becomes

$$\hat{r}(t) - \phi\mathbb{E}_t^X\left[\Omega(t)\frac{\psi(\tau,t)}{\hat{\psi}(\tau,t)}\right] = \hat{r}^*(t).$$

This equation is not satisfied if two maturities feature a different price jump. However, full asset spanning requires a different price jump at two maturities. This contradiction implies that even when the set of securities can provide

[34]This case can be solved via dynamic programming using aggregate debt at market values $\hat{B}$ as a state variable. $\hat{B}$ is defined as in (2.17) with pre-shock prices and debt profile. Equation (G.1) holds for every maturity, so given $\hat{B}$, it represents a family of first-order conditions for $f(\tau,t)$. The debt profile then is associated with an insurance cost of $\dot{\hat{B}}$.

[35]The proof is a particular case of the one presented in Appendix C.8 for the case with default.

insurance in non-default states, the government's solution with commitment does not adopt a perfectly insuring scheme. The distortion follows because the echo-effect acts differently than the risk premium, it distorts valuations but not prices—we can see that even under risk-neutrality.

Default allows some hedging. Consider the case of only income shocks. Without default, we noted that there was no hedging role for maturity but now we show that with default, there is a role. Default opens the possibility of a partial hedging because prior to the shock, different maturities are priced differently. Post-shock prices are always $\psi(\tau, t^o) = 1$. This means that once a shock hits, the government can exploit the change in the yield curve to obtain capital gains in its portfolio. The change in the risk premium is akin to the spanning effect of an interest-rate jump.

General case. The option to default interrupts insurance across non-default states, but allows price variation even without interest-rate risk. As long as the cardinality of shocks is discrete, the maturity profile is indeterminate—a formal proof is found in Appendix C.8. One extreme case of indeterminacy is that of a shock which does not produce a jump in income nor interests, but only grants a default option. Aguiar et al. (Forthcoming) studies that shock in a discrete-time model similar to ours but without commitment.

H Additional figures

In this section we introduce the figures corresponding to the different exercises performed in the case of a $T \to \infty$, risk-neutral governments or turning off the revenue-echo effect ($\Omega = 0$).

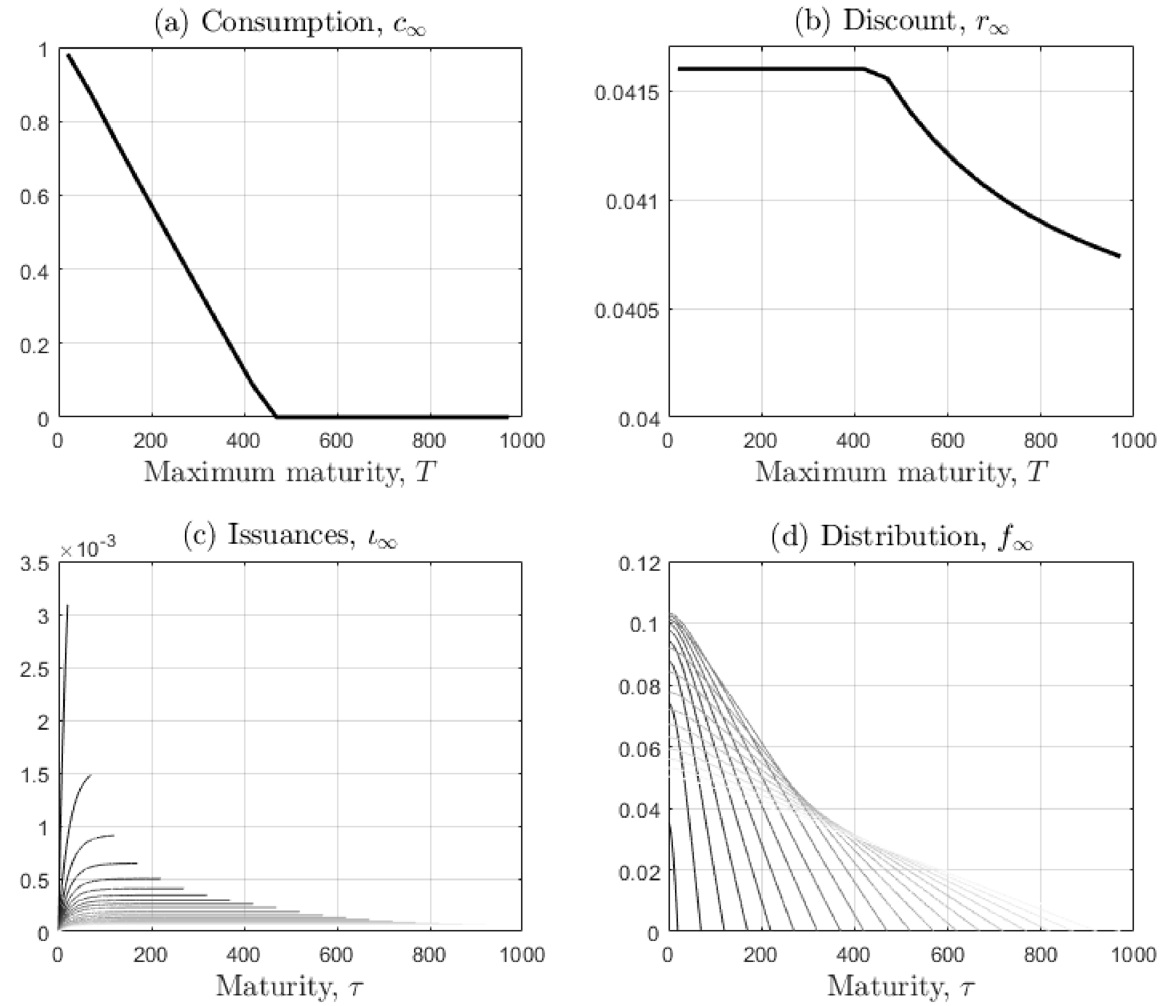

Figure H.1: Asymptotic equilibrium objects as a function of the maximum maturity (T).

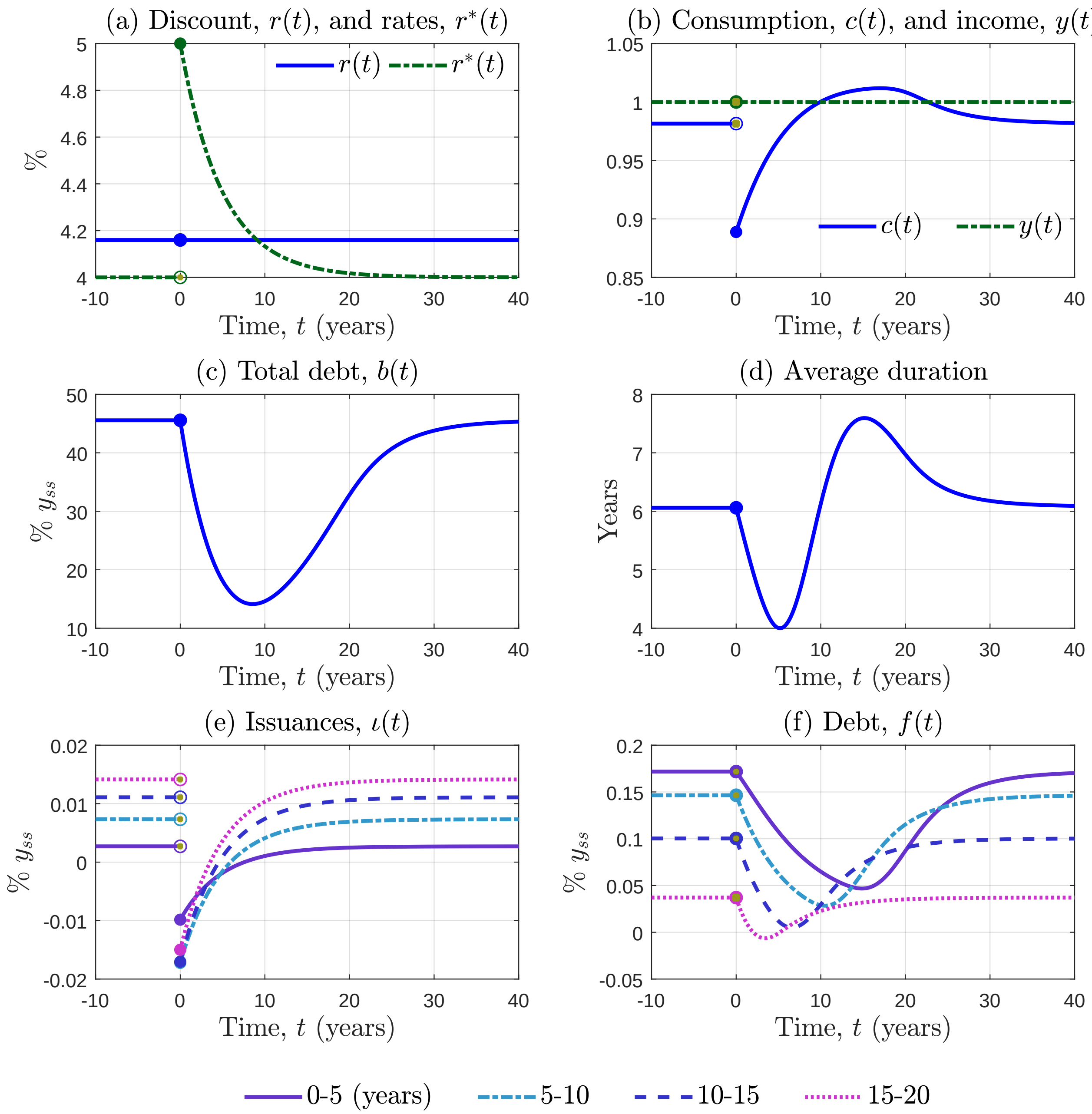

Figure H.2: Response to an unexpected shock to interest rates with $\sigma = 0$.

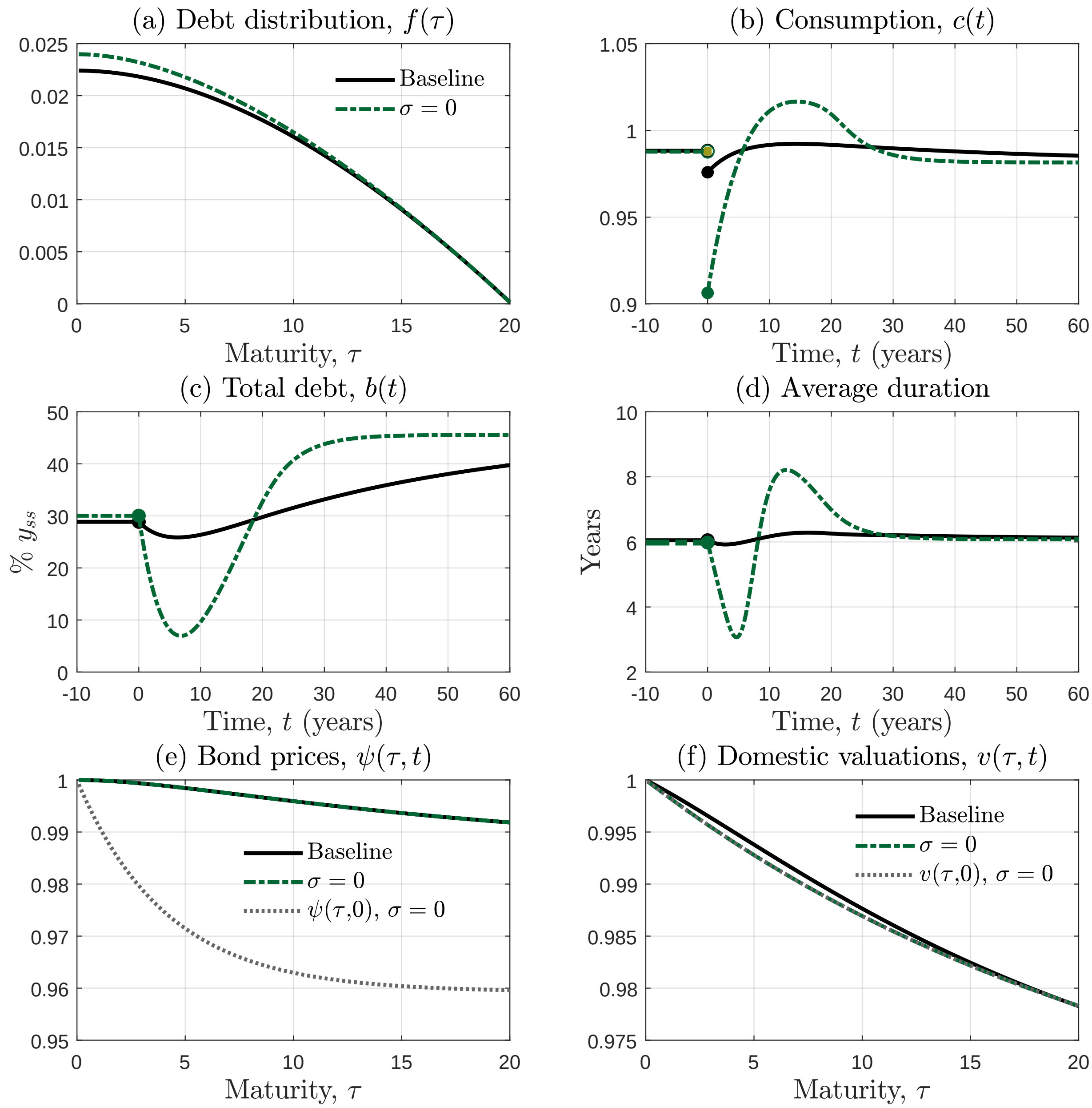

Figure H.3: Response to a shock to interest rates with $\sigma = 0$. 'Baseline' stands for the model starting at the baseline RSS and '$\sigma = 0$' for the case with a risk-neutral government.

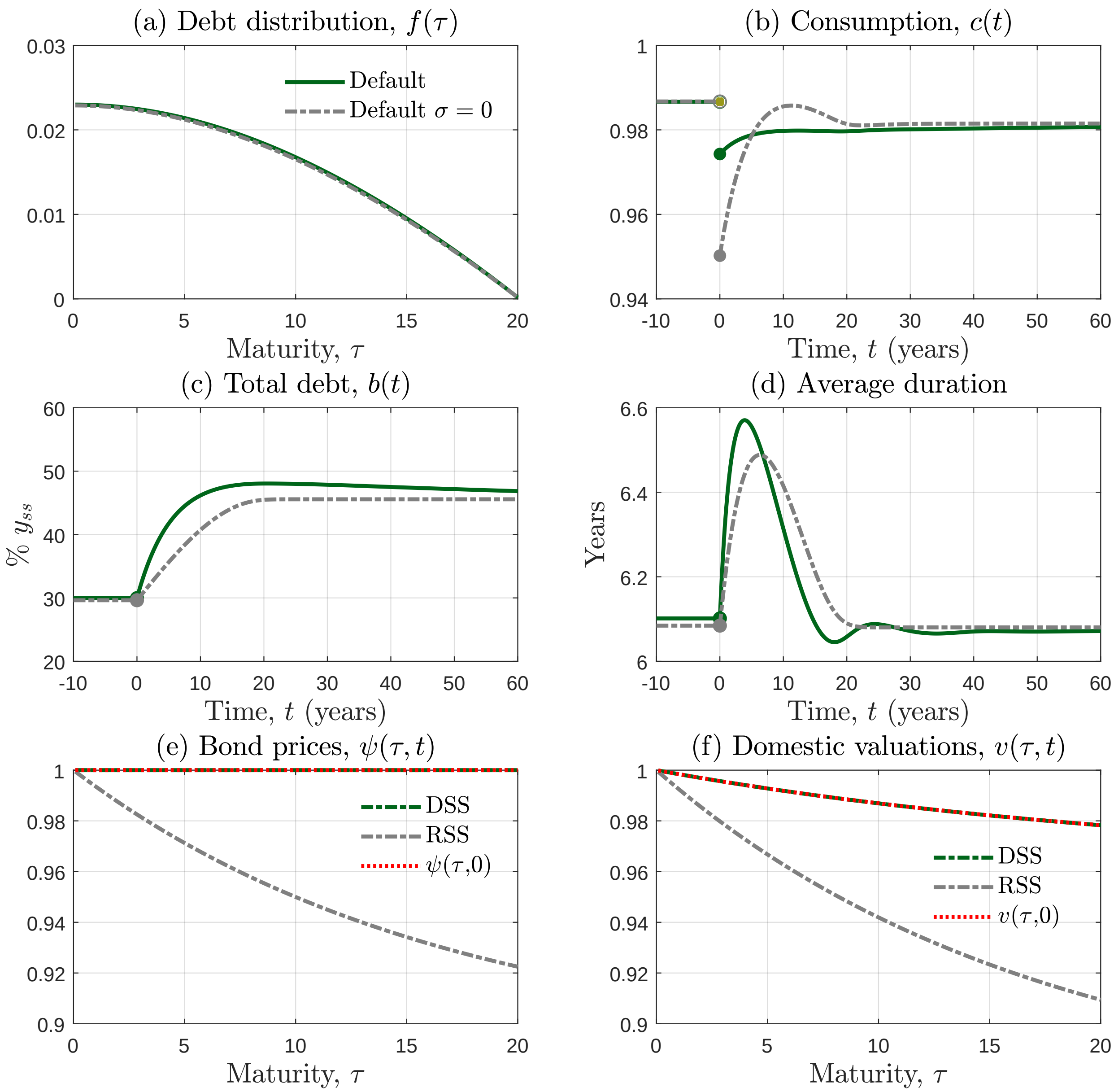

Figure H.4: Response to a shock to income with $\sigma = 0$ when the option to default is available. Panels (e) and (f) refer to the case with $\sigma = 0$.