*The Comparative Statics of Sorting**

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June 3, 2022

Abstract

We create a general and tractable theory of increasing sorting in pairwise matching models with transferable utility. Our partial order, *positive quadrant dependence*, subsumes Becker (1973) as the extreme cases with most and least sorting. It implies sorting by correlation of matched partners, or distance between partners. Our theory turns on *synergy* — the cross partial difference or derivative of match production. This reflects basic economic forces: diminishing returns, technological convexity, insurance, and match learning dynamics.

We prove that sorting increases if match synergy globally increases, and is also cross-sectionally monotone or single-crossing. Our theorems shed light on major economics sorting papers, affording immediate proofs and new insights. They open the door to fast predictions for new pairwise sorting models in economics.

*We have profited from comments in seminars at Harvard/MIT, Northwestern Kellogg, Princeton, the NBER, Berkeley, Michigan State, ASU, in particular, Hector Chade. Lones thanks the NSF for funding.

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1 Introduction

Assortative matching is the allocational theme in the vast literature on decentralized matching. This finding has seen application in marriage, employment, partnerships, optimal assignment, and pairwise trade. Becker (1973) showed that it emerges when match types are complementary. The power of this conclusion is also its weakness — higher "men" match with higher "women," *without exception*. Since it is an ideal, how should we understand sorting deviations? For instance, with search frictions, Shimer and Smith (2000) deduced sorting under stronger complementarity assumptions. But they found that matching set are centered about Becker's frictionless match partner.

While search and information frictions undeniably distort sorting, match productivity is surely the major driving force for the *changes in who matches with whom*. How does sorting vary across economic environments? Currently, our only general matching theory is Becker's — but it allows just two conclusions: either positive or negative assortative matching. His premises are also restrictive: a globally positive or globally negative cross partial difference or derivative. Chade, Eeckhout, and Smith (2017) explore many natural and some well-cited economic matching settings where both assumptions fail. The lack of a predictive general theory that applies to such cases has greatly limited the analytic reach of the matching literature in economics. Theory papers have all explicitly solved for the matching, and this too has focused excessive attention on the extreme solvable cases of perfect positive or negative sorting.

This paper fills this void: We develop a tractable general theory of sorting changes in the frictionless pairwise matching model with either finitely many or a continuum of types and *transferable utility* (TU). *We provide comparative statics predictions for the planner's problem, or equivalently, equilibria, without ever solving for either.* We shed light on influential economics matching papers since Becker (1973) without everywhere complementary types, and relax this major match payoff restriction for future work.

We argue that the *positive quadrant dependence* (PQD) partial order captures an economic meaning of "more assortative" (**Lemma 1**). This stochastic order ranks matching measures by the mass in the southwest quadrant. Intuitively, rising in the PQD order pushes matching mass closer to the diagonal. Rising in the PQD order, (*i*) the average distance between matched types falls, (*ii*) the correlation of matched types increases, *and* (*iii*) the regression coefficient of women on their partners' types increases. In other words, our sorting comparative statics findings are of direct empirical relevance in economics. By contrast, we show in §3.2 that *no coherent sorting theory can emerge based on covariance, correlation, or average distance between partners.*

To illustrate the PQD order, consider the six possible complete matchings among three men and three women (Figure 1). Each man matches with a weakly closer partner

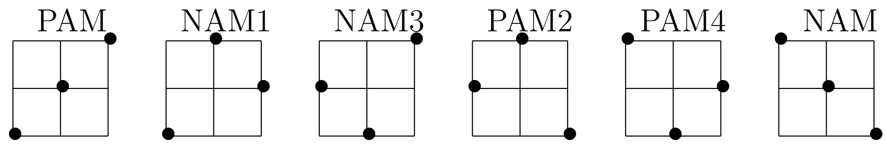

Figure 1: **Pure Matchings with 3 Types.** The possibilities are: negative and positive assortative matching (NAM and PAM), negative sorting in quadrants 1 and 3 (NAM1 and NAM3), and positive sorting in quadrants 2 and 4 (PAM2 and PAM4).

in PAM than in NAM1 or NAM3, in turn each closer than in PAM2 or PAM4, and finally than in NAM. Meanwhile, the matchings NAM1 and NAM3, as well as PAM2 and PAM4, are incomparable. We thus have a partial order:

$$\text{PAM} \succ_{PQD} [\text{NAM1, NAM3}] \succ_{PQD} [\text{PAM2, PAM4}] \succ_{PQD} \text{NAM} \tag{1}$$

We introduce an assumption on production functions that is a local version of Becker's complementarity. *Synergy* is the cross partial difference of production with finitely many types, and the cross partial derivative with continuous types. Thus synergy is everywhere positive for supermodular functions, and everywhere positive for submodular functions. To highlight its central role, we derive a formula rewriting total match output (4) as a weighted average of match synergy. This means that any matching characterization must turn on synergy. Becker (1973) deduces positive sorting for globally positive synergy, and negative sorting for globally negative synergy. We subsume a vast array of intermediate cases, where synergy changes sign, and so greatly expand the reach of matching theory.

Rephrasing Becker (1973), globally positive synergy implies assortative matching. So then is sorting greater with more synergistic production? A simple three-type example refutes this conjecture — the optimal matching oscillates between NAM1 and NAM3 as synergy rises in Figure 4. So on the one hand, an increasing sorting theory must build on production synergy, but on the other, sorting need not increase even if synergy everywhere does. This highlights the difficulty of our comparative statics goal.

While increasing synergy is not enough for increasing sorting, **Proposition 0** finds that sorting cannot fall in the PQD order when synergy globally weakly rises. This argument exhausts the strength of monotone comparative statics logic in the matching setting, and allows unranked oscillations, like NAM1 to NAM3, as synergy rises.

To secure increasing sorting, we need stronger assumptions. We add in cross-sectional restrictions on synergy. Our easiest to state such result is **Proposition** 1. It says that sorting increases if synergy everywhere increases, and is cross-sectionally monotone in partner types. Our theory builds on this, relaxing the cross-sectional monotonicity, since synergy is not monotone in typical matching models.

Our key increasing sorting result in the paper is **Proposition** 2. It weakens both the time series and cross-sectional premises of Proposition 1, replacing monotonicity conditions with weaker sign change provisos. Specifically, the new time series requirement is that the total synergy aggregated on unions of rectangular partner sets changes sign only from negative to positive. The new cross-sectional premise is that the total synergy on rectangular sets changes sign just once as it shifts north-east.

Next, **Proposition** 3 replaces the cross-sectional premise of Proposition 2 with an assumption on marginal rectangular synergy. We strengthen the cross-sectional premise one further step in **Proposition 4**, and arrive at our easiest to check general sorting result, ideal for continuum type matching models. It posits that synergy changes sign only from negative to positive, both over time and cross-sectionally. But this is not enough, since single crossing is not preserved under addition. To ensure valid aggregation, we introduce a new *proportional upcrossing* condition. This ensures that positive synergy rises proportionately more than absolute negative synergy.

Finally, the logic of the paper is that Proposition 2 implies Proposition 3 implies Proposition 4 implies Proposition 1. We prove Proposition 2 for finitely many types. The proof in §C.1 by induction on the number of types is a key contribution of the paper. Notably, it never solves for an optimum. Rather it chases down failures of the comparative static to the possible shift from NAM to the n-type version of NAM3.

Distributional shifts can also greatly impact sorting: A rise in high types of women may have profound macro consequences on sorting. **Corollary 2** shows how our increasing sorting theory applies when the type distribution shifts. For such distribution shifts can be equivalent to productive synergistic shifts, so that our theory applies.

To see how much we expand the predictive reach of matching theory, assume 100 men and 100 women. Becker (1973) applies for two synergy sign combinations. We encompass $2 \cdot 99^2$ sign combinations — and ones that specifically arise in applications.[1]

Economic Applications of Our Theory. Becker's work has sparked a vast literature on the transferable utility matching paradigm. But his sorting conclusion requires complementary partner types, which fails in many recent matching models.

We argue in §7 that our theory sweeps in many old and new economic models:

1. The typical economic force of *diminishing returns* lowers synergy and so sorting.
2. Match synergy is greater for *"weakest link" technologies* — namely, where the lesser type impacts payoffs more. We argue that this force formally makes the technology more convex in types — just as $\min(x, y)$ is more convex than $x + y$.

[1]For our upcrossing assumption, a sign change can occur after any of 99 men and 99 women.

3. The opposite case of a *"strongest link"* matching technology captures insurance — to wit, the greater type matters more. These technologies are the least synergistic. For consider the principal-agent matching model. Serfes (2005) found that negative sorting — more risk averse agents with safer projects — arises for a low effort disutility, but positive sorting for a high disutility. Our theory allows a quick stronger characterization that sorting rises in the disutility of effort.
4. Our theory also speaks to dynamic matching with evolving types. In a model of *mentor-protege workplace learning*, matching with a better mentor improves the protege's future type. This strongest link technology lowers match synergy.[2]

Our paper is related to a math literature on the PQD order. Lehmann (1966) introduced the PQD order, and showed that several common measures of correlation are weakly positive for any matching that is PQD higher than uniform random matching. Cambanis, Simons, and Stout (1976) found that total output weakly rises when the matching shifts up in the PQD order whenever synergy is everywhere non-negative. A corollary of this result is that sorting cannot fall in the PQD order when synergy globally increases. Techen (1980) showed that non-negative synergy is necessary for total output to rise for any upward shift in the PQD order.

Longer proofs and new monotone comparative statics results are in Appendices.

2 Becker's Marriage Model and Planner's Result

Our model is standardly adapted from Becker and the pairwise matching literature with two groups (men and women, firms and workers, buyers and sellers) or one (partnership model). To subsume both finite and continuum type models, we posit a unit mass of "women" and "men" with respective *types* $x, y \in [0,1]$ and cdfs G and H. We assume absolutely continuous type distributions G and H, and for the finite type model, G and H are discrete measures with equal weights on female types $0 \leq x_1 < x_2 < \cdots < x_n \leq 1$ and male types $0 \leq y_1 < y_2 < \cdots < y_n \leq 1$. In the finite types case, assume equal sized women and men relabeled as $i, j \in \{1, 2, \ldots, n\}$, respectively.

We assume a C^2 production function $\phi > 0$, so that types x and y jointly produce $\phi(x, y)$. In the finite type model, the output for match (i, j) is $f_{ij} = \phi(x_i, y_j) \in \mathbb{R}$. Production is *supermodular* or *submodular* (SPM or SBM) if for all $x' < x''$ and $y' < y''$:

$$\phi(x', y') + \phi(x'', y'') \geq (\leq)\ \phi(x', y'') + \phi(x'', y') \tag{2}$$

[2]Bayesian updating need not inherit supermodularity in Anderson and Smith (2010). Supermodularity is often not preserved in our work with evolving human capital (Anderson and Smith, 2012).

Strict supermodularity (respectively, strict SBM) asserts globaly strict inequality in (2). And production is *modular* (or additive) when (2) always holds with equality.

Since output is positive, everyone matches — even if allowed not to. A *matching* is a bivariate cdf $M \in \mathcal{M}(G,H)$ on $[0,1]^2$ with marginals G and H. In the finite type case, G and H put equal unit weight on $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$. A finite matching is a nonnegative matrix $[m_{ij}]$, with cdf $M_{i_0 j_0} = \sum_{1\le i\le i_0, 1\le j\le j_0} m_{ij}$, and unit marginals $\sum_i m_{ij_0} = 1 = \sum_j m_{i_0 j}$ for all men i_0 and women j_0. In a *pure matching*, $[m_{ij}]$ is a matrix of 0's and 1's, with everyone matched to a unique partner.

There are two perfect sorting flavors. In *positive assortative matching* (PAM), any woman type of x at quantile $G(x)$ pairs with a man of type y at the same quantile $H(y)$, and thus the match cdf is $M(x,y) = \min(G(x), H(y))$. In *negative assortative matching* (NAM), complementary quantiles match, and so $M(x,y) = \max(G(x) + H(y) - 1, 0)$. Matched types are *uncorrelated* given uniform matching, and so $M(x,y) = G(x)H(y)$.

The *partnership* (or unisex) model is a special case where types x and y share a common distribution, $G = H$, the production function ϕ is symmetric ($\phi(x,y) = \phi(y,x)$), and so too is the optimal matching distribution $M(x,y) \equiv M(y,x)$. In this case, PAM simply reduces to the matching $y = x$, or match with the same type.

A social planner maximizes total match output, namely $\sum_{i=1}^n \sum_{j=1}^n f_{ij}(\theta) m_{ij}$ with finite types, or more generally $\int_{[0,1]^2} \phi(x,y|\theta) M(dx,dy)$, where we index output $\phi(x,y|\theta)$ by a (*often suppressed*) *state* $\theta \in \Theta$, a partially ordered set. As optimal matching $\mathcal{M}^*(\theta)$ solves:

$$\mathcal{M}^*(\theta) = \arg \max_{M \in \mathcal{M}(G,H)} \int_{[0,1]^2} \phi(x,y|\theta) M(dx,dy) \tag{3}$$

After proving existence of $\mathcal{M}^*$, Gretsky, Ostroy, and Zame (1992) decentralize it as a competitive equilibrium, namely, the core of the matching game among women and men. One can also think of freely entering firms hiring workers x and capital y in competitive markets. *So, all our theory applies to equilibrium sorting in such markets.*

Problem (3) has been solved in just three general cases: All feasible matchings are optimal with additive production, while Becker solved for SBM and SPM production:[3]

Becker's Sorting Result. *Given SPM (SBM) production ϕ, PAM (NAM) is an optimal matching. Given strict SPM (SBM), these pairings are uniquely optimal.*

For an intuition, assume finitely many types and SPM (2). A maximum of (3) obviously exists. To see uniqueness, not that if ever women $x' < x''$ and men $y' < y''$ are negatively sorted into matches (x', y'') and (x'', y'), then total output is raised by rematching them as (x', y') and (x'', y''). A proof for any number of types is in §3.

[3] Koopmans and Beckmann (1957) decentralize the solution as a competitive equilibrium assuming TU. Legros and Newman (2007) show that some NTU models can be mapped into the TU paradigm.

Without SBM or SPM, solving the general social planner's problem (3) is a hard open question. We bipass this, and ask how the optimal matching $\mathcal{M}^*(\theta)$ changes in θ. We derive its comparative statics in θ when output $\phi(x, y|\theta)$ is neither SPM or SBM. Hereafter, a *time series* property suggestively refers to changes in the state θ,[4,5] and a *cross-sectional property* to production changes over the type space. We then apply our finding in several matching models across economics, without SPM or SBM output.

Throughout the paper, we present finite type and continuum type results together, as synergy is a common theme. We draw both intuition and our overall inductive proof logic from the finite type case, and derive the continuum type results by taking limits.

3 Synergy and Sorting Measurement

3.1 Synergy and the Positive Quadrant Dependence Order

We now introduce a local measure of Becker's restrictive assumption supermodularity. In finite type models, we suggestively call the cross partial difference of output *synergy*:

$$s_{ij}(\theta) = f_{i+1j+1}(\theta) + f_{ij}(\theta) - f_{i+1j}(\theta) - f_{ij+1}(\theta)$$

The central importance of synergy is revealed by expressing match output as a weighted sum of match synergies. Appendix §A proves the next double sum match output by parts:

$$\textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} m_{ij} = \sum_{i=1}^{n} f_{in} - \sum_{j=1}^{n-1} \left[f_{nj+1} - f_{nj}\right] j + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij} M_{ij} \qquad (4)$$

So any two production functions with identical synergies share the optimal matching. For instance, if production is linear, then synergy vanishes, and all match distributions yield the same output. Becker focused on the SPM case with globally nonnegative synergy. We henceforth link changes in the synergy to changes in the optimal matching.

Next, we introduce *positive quadrant dependence (PQD)*. This is a partial order on bivariate probability distributions $M_1, M_2 \in \mathcal{M}(G, H)$. Matching measure M_2 is *PQD higher than* M_1, or $M_2 \succeq_{PQD} M_1$, if $M_2(x, y) \geq M_1(x, y)$ for all types x, y. So M_2 puts more weight than M_1 on all lower (southwest) orthants. As M_1 and M_2 share marginals, M_2 puts more weight than M_1 on all upper (northeast) orthants too (Figure 2).

As noted in (1), PQD only partially orders the six possible pure matchings on three

[4] The term time-series is used to distinguish variation *across* matching markets from changes across types within a market. The state could also represent geographic differentiation in matching markets.

[5] Equivalently, our theory compares sorting for two production functions ϕ_1 and ϕ_2 (i.e. $\theta_1 < \theta_2$).

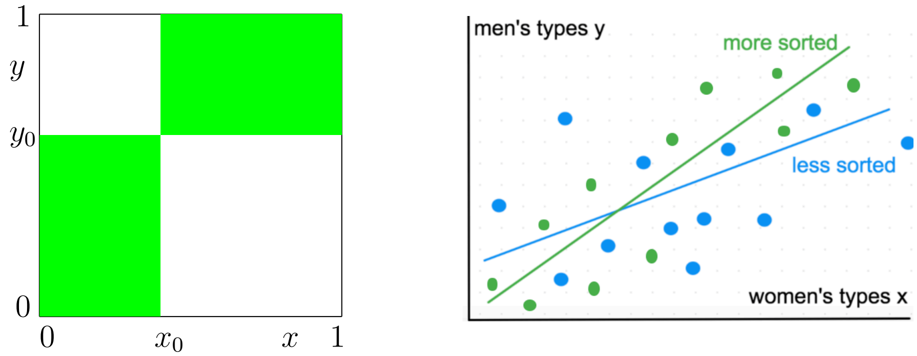

Figure 2: **PQD Order.** Left: PQD increases for cdfs on $[0,1]^2$ raise the probability mass on all lower left rectangles (corners $(0,0)$ and (x_0, y_0)), and so on all upper right rectangle (corners (x_0, y_0) and $(1,1)$). Right: We schematically depict Lemma 1(c).

types. In terms of Becker's bounds, match cdf's are sandwiched above NAM and below PAM:[6]

$$\max(G(x) + H(y) - 1, 0) \leq M(x,y) \leq \min(G(x), H(y)) \tag{5}$$

The second inequality says that the mass of matched men and women in $[0,x] \times [0,y]$ is at most the supply of men or women. The first inequality is more subtle — or $1 - M(x,y) \leq \min(1 - G(x) + 1 - H(y), 1)$, says the mass of men and women in $[0,x] \times [0,y]$ not matched is at most the supply of men plus the supply of women.

Becker's Result follows from the bounds (5) and either summation by parts formula (4), or the continuum analog in Lemma 3 in §D.2. For SPM output implies all $s_{ij} \geq 0$, and so by (4) output is highest when the cdf $M(x,y)$ is maximal. So PAM dominates all other matchings. Similarly, SBM implies globally nonpositive synergy, $s_{ij} \leq 0$, and thus output is highest when the match cdf $M(x,y)$ is minimal, namely, for NAM. More generally, the PQD and SPM orders coincide in $\mathbb{R}^2$, i.e. *increases in the PQD order increase (reduce) the total output for any SPM (SBM) function* ϕ:[7]

$$M_2 \succeq_{PQD} M_1 \quad \Leftrightarrow \quad \int \phi(x,y) M_2(dx,dy) \geq \int \phi(x,y) M_1(dx,dy) \qquad \forall \phi \text{ SPM} \tag{6}$$

The PQD sorting measure implies some more typical economically relevant measures for measured traits $u(x)$ and $v(y)$ of women x and men y, increasing in x and y:

Lemma 1. *Fix increasing functions u and v. Given a PQD order upward shift:*
(a) the average geometric distance $E[|u(X) - v(Y)|^\gamma]$ for matched types falls, if $\gamma \geq 1$;
(b) the covariance $E_M[u(X)v(Y)] - E[u(X)]E[v(Y)]$ across matched pairs rises;
(c) the coefficient in a linear regression of $v(y)$ on $u(x)$ across matched pairs rises.

[6] Cambanis, Simons, and Stout (1976) noted these bounds.

[7] Lehmann (1966) introduces the PQD order, and Cambanis, Simons, and Stout (1976) prove that the SPM order implies the PQD ranking in $\mathbb{R}^2$. Techen (1980) proves the converse.

Figure 3: **Sorting Statistic Disagreement.** Assume a uniform distribution on four types $\{1, 2, 3, 4\}$, with PQD incomparable matchings depicted. Covariance-based sorting statistics deem the circle matching more sorted (e.g. a higher correlation coefficient), while the bullet matching is more sorted by the average distance between partners.

Proof of (a): By inequality (6) it suffices that $|u(x) - v(y)|^\gamma$ is SBM for all $\gamma \geq 1$. Since $-\psi(u - v)$ is SPM for all convex ψ, by Lemma 2.6.2-(b) in Topkis (1998), we have $-|u - v|^\gamma$ SPM for all $\gamma \geq 1$. So, $|u(x) - v(y)|^\gamma$ is SBM for all increasing u and v.

Proof of (b): Since the marginal distributions on X and Y is constant for all $M \in \mathcal{M}(G, H)$, and $u(x)v(y)$ is supermodular for all increasing u and v, the covariance $E_M[XY] - E[X]E[Y]$ between matched types increases in the PQD order by (6).

Proof of (c): The coefficient $c_1 = cov(u(X)v(Y))/var(v(X))$ in the univariate match partner regression $v(y) = c_0 + c_1 u(x)$ increases in the PQD order, by part (b). $\square$

3.2 PQD is Superior to Common Sorting Measures

PQD is an *ordinal sorting ranking*, like PAM — not dependent on type sizes. So if educational sorting PQD rises, then this holds regardless of whether it is measured in highest degree, schooling years, etc. But for non-PQD comparable matching changes, the sorting conclusion can reverse if the choice of cardinal measure changes (Figure 4).

Lemma 1 established that a PQD increase implies commonly used sorting statistics increase. We now show that sorting predictions based on these common statistics depend on the scaling of types, and can easily move in *opposite* directions when the matching changes. This highlights why we use the stronger ordinal PQD sorting order.

Note that the *covariance* and correlation coefficient of matching partner types, and the linear regression coefficient of y on partner type x are co-monotone for matching changes, since each statistic is an increasing function of the other. So we consider the covariance sorting statistics and the average *distance* between match partner types. Easily, the partner covariance or distance depends on the type scaling, and may move in opposite directions as the matching changes. To see this assume three types, and consider a non-PQD comparable NAM1 to NAM3 change. If $x \in \{1, 2, 3\}$ and $y \in \{0.5, 1.8, 3\}$, then the covariance between matched types and average distance between partners both fall, i.e. sorting falls if measured by type correlation, but rises if measured by average distance between matched types. On the other hand, if $y \in \{0.5, 2.5, 5\}$, Match Payoffs

	x_1	x_2	x_3
y_3	9	**14**	18
y_2	5	2	**14**
y_1	**1**	5	9

$\to$

	x_1	x_2	x_3
y_3	9	16	**24**
y_2	**5**	3	16
y_1	1	**5**	9

$\to$

	x_1	x_2	x_3
y_3	9	**20**	30
y_2	5	6	**20**
y_1	**1**	5	9

$\to$

	x_1	x_2	x_3
y_3	9	22	**36**
y_2	**5**	7	22
y_1	1	**5**	9

Cross Partial Differences of Match Payoffs

	x_1x_2	x_2x_3
y_2y_3	8	-8
y_1y_2	-7	8

$\to$

	x_1x_2	x_2x_3
y_2y_3	9	-5
y_1y_2	-6	9

$\to$

	x_1x_2	x_2x_3
y_2y_3	10	-4
y_1y_2	-3	10

$\to$

	x_1x_2	x_2x_3
y_2y_3	11	-1
y_1y_2	-2	11

Figure 4: **Sorting Need Not Rise in Synergy.** Top: the unique efficient matching alternates between NAM1 and NAM3. Bottom: match synergies (cross payoff differences) strictly increase as we move right, but sorting does not PQD rise. Sorting by two common cardinal measures can move contrarily. If $x \in \{1, 2, 3\}$ and $y \in \{0.5, 1.8, 3\}$, *NAM1 to NAM3 shifts reduce both covariance* and *average distance between partners.*

match type correlation rises, and average distance between matched types falls. Both sorting measures fall when $y \in \{0.5, 2.5, 3\}$ and both rise when $y \in \{0.5, 2.5, 3\}$. So *any sign pattern is consistent with a NAM1 to NAM3 shift*.

If we convert to *quantile space* and restrict to three types, then the covariance and the average distance ranking coincides for (NAM1,NAM3) and (PAM2,PAM4). But equivalence fails with four types, as Figure 3 shows.

4 What Happens When Synergy Rises?

Since Becker shows that globally negative synergy leads to NAM, and globally positive synergy leads to PAM, one might surmise that sorting increases if synergy increases everywhere. This natural conjecture fails: In Figure 4, synergy strictly increases at each step, and yet the uniquely optimal matching oscillates between the non PQD-comparable NAM1 and NAM3. What goes wrong?

Firstly, our optimization does obey the single crossing property when synergy rises in θ. But monotone comparative statics is not an option, because the domain of matching cdf's is not a lattice with the PQD order (Müller and Scarsini, 2006). Indeed, NAM1 and NAM3 in (1) are both pure upper bounds for PAM2 and PAM4, but neither is least. More strongly, there is no least mixed least upper bound for PAM2 and PAM4.

Appendix D extends the theory of monotone comparative statics to our case with a single crossing condition, but not on a lattice domain. Specifically, for our matching context, say that *sorting is nowhere decreasing* in θ if the matching never falls in the PQD order. So for all $\theta_2 \succeq \theta_1$, if $M_1 \in \mathcal{M}^*(\theta_1)$ and $M_2 \in \mathcal{M}^*(\theta_2)$ are ranked

$M_1 \succeq_{PQD} M_2$, then we have $M_2 \in \mathcal{M}^*(\theta_1)$ and $M_1 \in \mathcal{M}^*(\theta_2)$.

Proposition 0. *Sorting is nowhere decreasing in θ if synergy is non-decreasing in θ.*

PROOF: By match payoff formulation (4), the payoff gain moving from matching M'' to matching M' is $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)(M'_{ij}-M''_{ij})$. Since $M' \succeq_{PQD} M''$ (namely, $M' \geq M''$), if $\theta'' \succeq \theta'$, then *the Planner's objective function obeys increasing differences in* (M, θ):

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta'')(M'_{ij} - M''_{ij}) \geq \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta')(M'_{ij} - M''_{ij})$$

Assume that M' is optimal at θ' and M'' at θ''. Then

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta')(M'_{ij} - M''_{ij}) \geq 0 \geq \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta'')(M'_{ij} - M''_{ij})$$

But then equality holds everywhere: Hence, M' is optimal at θ'' and M'' at θ'.[8] □

All told, the optimal matching cannot fall in the PQD order if synergy rises.[9] We succeed by adding *cross-sectional* assumptions on synergy, and prove increasing sorting by induction on the number of types, rather than monotone comparative statics.

5 Increasing Sorting

5.1 Strictly Monotone Synergy in Types

Say that *sorting increases* in θ if $M_2 \succeq_{PQD} M_1$ for all $M_1 \in \mathcal{M}^*(\theta_1)$, $M_2 \in \mathcal{M}^*(\theta_2)$ and $\theta_2 \succeq \theta_1$. To preclude increasing sorting failures as in Figure 4, we add cross-sectional discipline. First consider the simplest case: *synergy is (strictly) monotone in types* if synergy is either non-decreasing (increasing) or non-increasing (decreasing) in (x, y).

This cross-sectional assumption means that synergy either increases to the "north and east", or "south and west". One might presume that this simplifies the planner optimization enough that sorting trivially increases when synergies do. One reason this is not true is that as synergy rises, rematching occurs at a global level. So in Figure 5, *matching moves away from the diagonal in the north-east quadrant as synergy rises.*

[8] If the space of matching cdf's were a lattice, then the same steps prove that $M' = M_1 \vee M_2 \in \mathcal{M}^*(\theta_2)$ and $M'' = M_1 \wedge M_2 \in \mathcal{M}^*(\theta_1)$ for any $M_1 \in \mathcal{M}^*(\theta_1)$ and $M_2 \in \mathcal{M}^*(\theta_2)$, i.e. the set of of optimal matchings $\mathcal{M}^*(\theta)$ increases in the strong set order.

[9] For completeness, an online Appendix D.1 generalizes Proposition 0, deriving a more general theory of comparative statics on posets. We thank a referee for the proof of the following special case of this general theory. He derived it as a corollary of Cambanis, Simons, and Stout (1976).

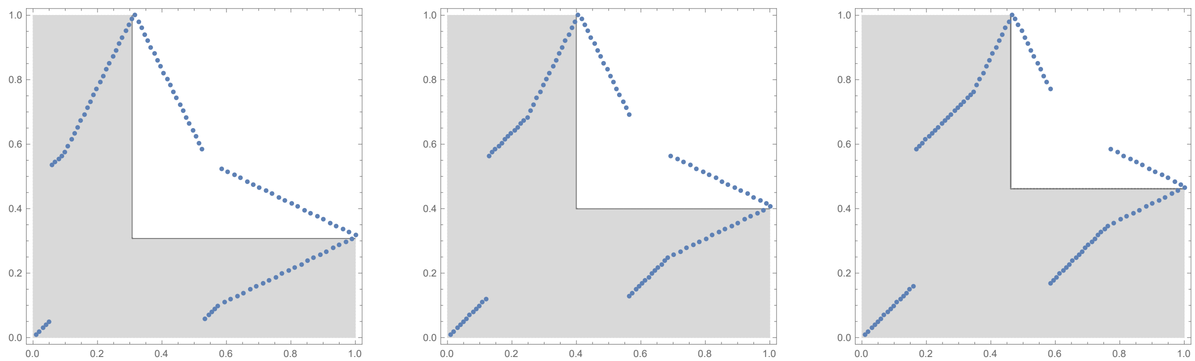

Figure 5: **Non-pure Matching Example for Proposition 1.** We numerically depict the matching support for the synergy function $\alpha - \beta \min\{x_i, x_j\}$. *All matching plots depict optimally matched (blue) pairs for a uniform distribution on a finite* 100×100 *matching array.* In each graph, synergy is positive (negative) on the shaded (unshaded) regions. Left to right plots assume $(\alpha, \beta) = (0.4, 1.3)$, $(0.4, 1)$, and $(0.6, 1.3)$.

Notably, this cross-sectional assumption is not so strong that it eliminates the partialness of the PQD order. For instance, PAM2 and PAM4 can both emerge as optimal matchings when synergy is strictly monotone in type (Figure 7, left). But if synergy is monotone in types, then the optimal matching cannot shift from PAM2 to PAM4 (or vice-versa) as synergy globally rises (Figure 7, right).

Proposition 1. *Assume synergy is non-decreasing in θ. Sorting is increasing in θ for: (a) generic finite type models if synergy is monotone in types and (b) continuum types model if synergy is strictly monotone in types.*

To illustrates this first sorting result, consider the production function $\phi = \alpha xy + \beta(xy)^2$. Then since synergy $\phi_{12} = \alpha + 2\beta xy$ is strictly increasing in (α, β), sorting rises in both parameters by Proposition 1. A more complicated matching pattern emerges for the synergy function $\phi_{12}(x, y) = \alpha - \beta \min\{x, y\}$ (recall that the optimal matching is fully determined by synergy (4)). Synergy is monotone in types (non-decreasing when $\beta \leq 0$ and non-increasing when $\beta \geq 0$), and increases in α and falls in β. By Proposition 1, sorting increases in α and falls in β (Figure 5). Increasing sorting emerges despite potential complexity of the optimal matching. For instance, the optimal matching is not simply described by a cutoff in the type space with PAM above, and NAM below, this cutoff (or vice versa). The matching here switches back and forth between locally positive and locally negative sorting. This finite type plots also suggests that the optimal matching need not be pure (one-to-one) in the continuum limit. But none of our continuum type sorting results require purity.

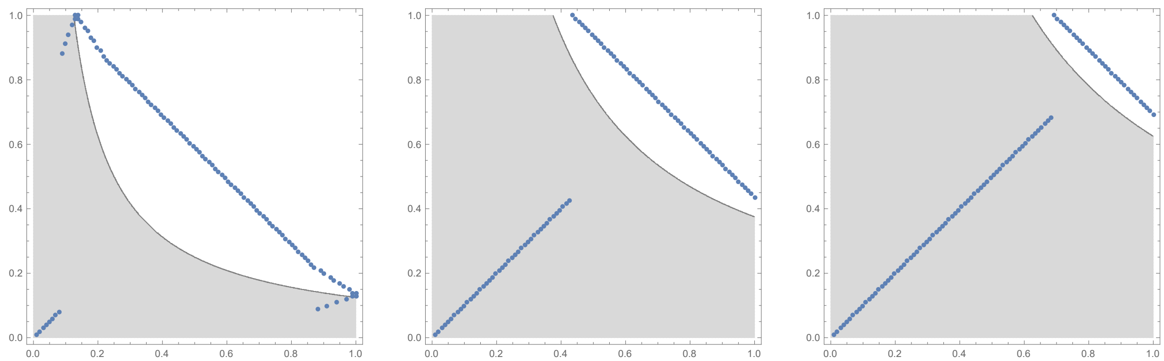

Figure 6: **Quadratic Production Example for Proposition 1.** We depict positive synergy (shade) and optimally matched pairs (dots) for a uniform distribution on 100 types and production $\phi = \alpha xy + \beta(xy)^2$. Sorting rises left to right in (α, β). The left plot assumes $(\alpha, \beta) = (0.5, -1)$, the middle $(\alpha, \beta) = (1.5, -1)$, the right $(\alpha, \beta) = (1.5, -0.6)$.

5.2 One-Crossing Rectangular Synergy in Types

While the conditions in Proposition 1 are quick to check, they do not hold in many economic applications. We instead derive and prove stronger result with a weaker and more commonly met single-crossing premise: A function Υ is *upcrossing in t*[10] on a partially ordered set T if $\Upsilon(t) \geq (>)0$ implies $\Upsilon(t') \geq (>)0$ for all $t' \succeq t$, *downcrossing in t* if $-\Upsilon$ is upcrossing, and *one-crossing in t* if it is upcrossing *or* downcrossing. Strict versions of these conditions require that weak inequalities imply strict inequalities. For example, Υ is *strictly upcrossing* if $\Upsilon(t) \geq 0$ implies $\Upsilon(t') > 0$, for all $t' > t$.

The *rectangle* $r \equiv (i_1, j_1, i_2, j_2) \in \mathbb{N}^4$ denotes two women $i_1 < i_2$ and men $j_1 < j_2$. *Rectangular synergy* $S(r|\theta) : \mathbb{N}^4 \to \mathbb{R}$ sums synergies $s_{ij}(\theta)$ inside the rectangle r:

$$S(r|\theta) \equiv \sum_{i=i_1}^{i_2-1} \sum_{j=j_1}^{j_2-1} s_{ij}(\theta) = f_{i_1 j_1}(\theta) + f_{i_2 j_2}(\theta) - f_{i_1 j_2}(\theta) - f_{i_2 j_1}(\theta) \tag{7}$$

This is the gain on rectangle r to positively sorting (creating couples $(i_1, j_1) < (i_2, j_2)$) versus negatively sorting (creating couples (i_1, j_2) and (i_2, j_1)). For a type continuum, $S(R|\theta) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \phi_{12}(x, y|\theta) dx dy$ for $R = (x_1, y_1, x_2, y_2)$.

Summed rectangular synergy adds synergy on any finite set of disjoint rectangles: $\{r_k\}$ with finite types, or $\{R_k\}$ with a continuum of types.[11] So equipped, our key time-series assumption asserts that summed rectangular synergy is upcrossing in θ. Easily, summed rectangular synergy is upcrossing in θ if synergy is non-decreasing in θ.

[10] The single crossing property usually implies a two dimensional functional domain. To avoid this confusion, and clarify the direction, we instead use the suggestive terms *upcrossing* and *downcrossing*.

[11] The proof only needs this assumption for sums of rectangles sharing a common northeast corner.

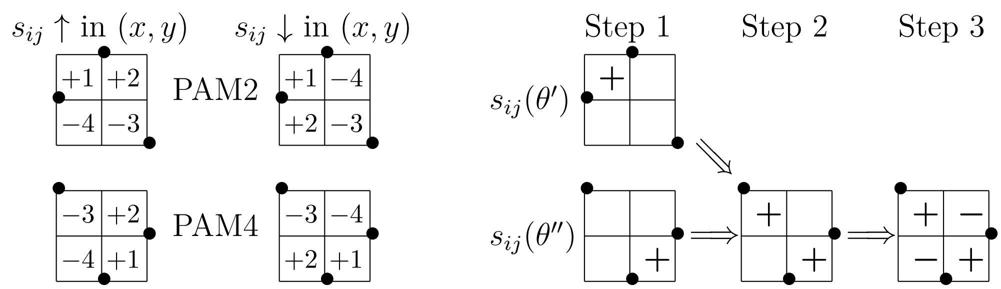

Figure 7: **The Role of our Cross-Sectional Synergy Assumption.** At left, we show that even strictly monotone synergy in types still allows PAM2 and PAM4, and so PQD is still a partial order on allowable matchings. At right, we show how the weaker upcrossing synergy assumption precludes non-PQD ranked matching shifts. Assume instead PAM2 optimal at θ' and PAM4 at θ''. Given the implied synergy signs in Step 1, Step 2 deduces $s_{12} > 0$ from upcrossing synergy as $\theta' \to \theta''$. Next, rectangular synergy is negative on the top row and left column rectangles at θ', by NAM. Step 3 fully signs all synergies, and violates up or down crossing. Contradiction.

Our first cross-sectional assumption uses the northeast partial order on rectangles: $r \succeq_{NE} r'$, if diagonally opposite corners of r are weakly higher than r'. Rectangular synergy is *one-crossing* in types if $S(r|\theta)$ is upcrossing (downcrossing) in r, for all θ.

Proposition 2. *Assume (A1)* summed rectangular synergy is upcrossing in θ *and (B1)* rectangular synergy is one-crossing in types. *If there is a unique optimal matching at* $\theta_2 \succ \theta_1$*, then sorting is PQD higher at* θ_2 *than* θ_1*.*

Proposition 2 is our key result. That rectangular synergy is one-crossing in types precludes the example in Figure 4, with nonmonotone sorting. For finite type models, the optimal matching is generically unique by Koopmans and Beckmann (1957). We prove uniqueness in the continuum type model in §C.3 by transportation theory.

Our proof in Appendix §C.1 is by induction on the number of types. First, we show that uniquely optimal matchings are pure, with men partners $\mu = (\mu, \ldots, \mu_n)$ of women, or women partners $\omega = (\omega_1, \ldots, \omega_n)$ of men. We then WLOG, posit rectangular synergy upcrossing in types. Consider uniquely optimal matchings μ'' and μ' at states $\theta'' \succ \theta'$. The proof uses two key ingredients (Facts 2 and 5 in Appendix §C.1):

- The *premises* of the $(n+1)$-type model hold in an induced n-type model:

Ordered Couple Removal. Assume $n+1$ types, with rectangular synergy upcrossing in types at θ' and θ'' and upcrossing in states from θ' to θ''. Then both remain true in the n type model if we remove the same woman and a weakly higher man at θ', or the same man and a weakly higher woman at θ' — i.e. removing couple (i', j') at state θ' and couple (i'', j'') at state θ'', where $\langle i' = i''$ and $j' \geq j''\rangle$ or $\langle j' = j''$ and $i' \geq i''\rangle$.

	θ'			θ''		
	x_1x_2	x_2x_3	x_3x_4	x_1x_2	x_2x_3	x_3x_4
y_3y_4	-2	0	1	-1	1	2
y_2y_3	-4	-3	0	-3	-2	1
y_1y_2	-5	-4	2	-4	-3	-1

	θ'		θ''	
	x_1x_2	x_2x_4	x_1x_2	x_2x_4
y_3y_4	-2	**1**	-3	**−1**
y_1y_3	-9	-11	-4	-4

Figure 8: **Ordered Couple Removal Fail.** Start with synergies for $n = 4$ types that obey the premise of Proposition 2 as shown on the left. Removing woman x_3 and man y_2 at θ' and woman x_3 and the higher man y_4 at θ'' induces synergies for the remaining $n = 3$ types at right. Note that the red bold synergies are not upcrossing in θ.

The cross sectional assumption still holds *for any couple removal* because it reduces the number of rectangular synergies and thus reduces the number of inequalities to check. But the time series conclusion exploits the couple ordering. Figure 8 shows how this conclusion may fail if the removed couples are not ordered as stated.

- The *conclusion* of the n-type model is preserved in an induced $(n+1)$-type model:

PQD Preservation. Assume n-types, with ranked matchings $\hat{\mu} \succeq_{PQD} \mu$. Then adding matched couples, now indexed as either $(1, \hat{j}) \leq (1, j)$ or $(\hat{i}, 1) \leq (i, 1)$, preserves the PQD order in the $(n + 1)$-type model. So we preserve the PQD order if we add a new woman 1 with a lower male partner at $\hat{\mu}$ than μ, or a new man 1 with a lower woman partner at $\hat{\mu}$. Note: the newly matched couple is more sorted under $\hat{\mu}$ than μ.

The following three type version of our induction proof captures the key elements of the proof in the Appendix using these two building blocks. First, we argue:

A. Sorting rises in θ in all 2×2-type subsets, if rectangular synergy upcrosses in θ.

B. If rectangular synergy upcrosses in types, then neither μ' nor μ'' are NAM1. For rectangular synergy upcrossing in types precludes $s_{11} + s_{12} > 0 > s_{22}$ (Figure 9), as required if NAM1 is uniquely optimal. This holds for *any* 3×3 *subset of* $n \times n$ *types.*

C. Partners of woman 1 and man 1 each rise by one if the matching does not weakly PQD rise. This corresponds to Step 3 in the Appendix. Indeed, shifting from θ' to θ'':

(i) The partners of woman 1 and man 1 cannot rise by two (or more). For if so, then $1 = \mu_1' < \mu_1'' = 3$. Let man 3 be paired with woman $i > 1$ at θ', who is matched to man $j < 3$ at θ''. Remove couples $(i, 3)$ at θ' and (i, j) at θ''. Since these are ordered couples, synergy is upcrossing in θ in the two type model, by Item A. Thus, the matching in the induced two type model is higher at θ'' than θ', by Item C. But this is impossible, since woman 1 pairs with man 1 at θ' but woman 1 pairs with (the new) man 2 at θ''.

(ii) The partner of woman 1 strictly rises. Assume instead that her partner weakly falls from k to j. Remove couple $(1, k)$ at state θ' and couple $(1, j)$ at state θ''. As ordered

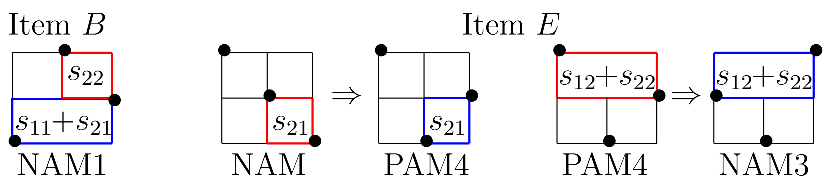

Figure 9: **Illustrations for 3-type Proof of Proposition 2.** Item B shows that NAM1 (left) is impossible when synergy is upcrossing in types. Mapping from NAM to PAM4 changes the payoff by s_{21} (Item E, left), while mapping from PAM4 to NAM3 changes the payoff by $s_{12} + s_{22}$ (Item E, right).

couples, the matching is higher at state θ'' than θ', by Ordered Couple Removal. Then adding couple $(1, k)$ and couple $(1, j)$ to the optimal two type matchings under θ' and θ'' preserves the PQD ordering, by PQD Preservation. So if the matching fails to weakly rise in the PQD order, then woman 1's partner cannot weakly fall.

Combining (i) and (ii), woman 1's partner increases by one.

D. If the matching does not weakly PQD rise, then it falls from NAM3 to NAM. By Item C, woman 1 cannot pair with man 3, nor man 1 with woman 3, at θ'. For, e.g., in the first case, by Item C, woman 1 matches with the nonexistent man 4 under θ''.

But woman 1 and man 1 cannot match at θ'. For if so, there are only two possible matchings for μ': either types 2 and 3 positively sort, and so $\mu' =$ PAM, or they do not, whence $\mu' =$ NAM1. Since Item A precludes NAM1, assume $\mu' =$ PAM. As the lowest two types match at θ', by Item C, woman 1 pairs with man 2 and man 1 with woman 2 at θ''. All told, the lowest two types positively sort at θ' and negatively sort at θ'' — violating rectangular synergy upcrossing in θ.

Now consider the remaining case: woman 1 pairs with man 2, and man 1 with woman 2, at θ'. Having matched the two lowest men and women, woman 3 must match with man 3. Altogether, μ' is NAM3 — namely, couples $\{(1, 2), (2, 1), (3, 3)\}$. By Item C, woman 1 matches with man 3, and man 1 with woman 3 at θ''. But then, the remaining man 2 and woman 2 match, i.e. μ'' is NAM: $\{(1, 3)(2, 2), (3, 1)\}$.

Item D captures Steps 4–7 in the induction proof. The next item distills Step 8:

E. The matching cannot fall from NAM3 to NAM. As seen in Figure 9, one can switch from NAM3 to NAM, by first moving to PAM4, then to NAM. The first shift rematches couples $(2, 2)$ and $(3, 1)$, into $(2, 1)$ and $(3, 2)$, changing output by synergy s_{21}. The second switch to NAM3 rematches couples $(1, 3)$ and $(3, 2)$ into $(1, 2)$ and $(3, 3)$, changing output by the synergy sum $s_{21} + s_{22}$. Combining these two swaps, we see that the NAM3 payoff exceeds the NAM payoff by synergy sum $s_{12} + s_{21} + s_{22}$. Since NAM3 is uniquely optimal for θ', and NAM uniquely optimal for θ'', we have

$$s_{12}(\theta'') + s_{21}(\theta'') + s_{22}(\theta'') < 0 < s_{12}(\theta') + s_{21}(\theta') + s_{22}(\theta')$$

	x_1	x_2	x_3
y_3	6	6	**11**
y_2	**4**	6	6
y_1	0	**4**	6

$\rightarrow$

	x_1	x_2	x_3
y_3	**7**	6	11
y_2	4	**6**	6
y_1	0	4	**7**

	x_1x_2	x_2x_3
y_2y_3	-2	5
y_1y_2	-2	-2

$\rightarrow$

	x_1x_2	x_2x_3
y_2y_3	-3	5
y_1y_2	-2	-3

Figure 10: **Falling Matching with Rectangular Synergy Upcrossing in Types and θ.** The unique efficient matching falls from NAM3 to NAM as θ' shifts up to θ''. The sorting premium S is upcrossing in rectangles r for each θ, and the signs of $S(r|\theta')$ and $S(r|\theta'')$ coincide for all r; thus, S is upcrossing from θ' to θ''. But Proposition 2 does not apply, as total synergy falls from 1 to -1 for the set that only excludes s_{11}.

This contradicts summed rectangular synergy upcrossing in θ.

Items D and E together imply that the matching weakly rises from θ' to θ''. □

Item E is the only place we exploit upcrossing *summed* rectangular synergy in θ, essential for our result. For in Figure 10, rectangular synergy is upcrossing in types and θ, and yet the uniquely optimal matching falls from NAM3 to NAM as θ rises.

5.3 One-Crossing Marginal Rectangular Synergy in Types

We now provide a stronger, but easier to check, cross-sectional assumption to deliver increasing sorting. Specifically, the *x-marginal rectangular synergy* $\Delta_i(i|j_1, j_2)$ is the sum of synergy over men in the interval $[j_1, j_2 - 1]$ and the *y-marginal rectangular synergy* $\Delta_j(j|i_1, i_2)$ is sum of synergy over women in the interval $[i_1, i_2 - 1]$, i.e.:

$$\Delta_i(i|j_1, j_2, \theta) \equiv \sum_{j=j_1}^{j_2-1} s_{ij}(\theta) \qquad \text{and} \qquad \Delta_j(j|i_1, i_2, \theta) \equiv \sum_{i=i_1}^{i_2-1} s_{ij}(\theta)$$

For a type continuum, the marginal rectangular synergy is an integral $\Delta_x(x|y_1, y_2, \theta) \equiv \int_{y_1}^{y_2} \phi_{12}(x, y)dy$ or $\Delta_y(y|x_1, x_2, \theta) \equiv \int_{x_1}^{x_2} \phi_{12}(x, y)dx$. These sums and integrals are *one-crossing* if they are respectively both upcrossing or both downcrossing in x and y.

Proposition 3. *Assume (A1)* summed rectangular synergy is upcrossing in θ *and (B2)* marginal rectangular synergy is one-crossing. *Sorting rises in θ in generic finite type, or continuum, types models with a strictly one-crossing marginal rectangular synergy.*

In §C.3, we integrate one-crossing marginal rectangular synergy (B2) to deduce one-crossing rectangular synergy (B1). Hence, *Proposition 2 implies Proposition 3.*

Next, we apply optimal transport theory to establish that the continuum optimal matching is unique when marginal rectangular synergy is strictly one-crossing.

5.4 Purely Local Assumptions on Synergy

We now develop a local theory of increasing sorting. A new assumption ensures that the one-crossing synergy aggregates to rectangles. We take inspiration from log-supermodularity, which is preserved by integration (Karlin and Rinott, 1980). Denote the positive and negative parts $f^+ \equiv \max(f,0)$ and $f^- \equiv -\min(f,0)$ of a function f. With a continuum of types, *synergy is proportionately upcrossing* if:

$$\phi_{12}^-(z \wedge z', \theta)\phi_{12}^+(z \vee z', \theta') \geq \phi_{12}^-(z, \theta')\phi_{12}^+(z', \theta) \tag{8}$$

for $z=(x,y)$, $z'=(x',y')$, and $\theta' \succeq \theta$, where meet $\wedge$ and join $\vee$ assume the vector order.

For a finite number of types, synergy is proportionately upcrossing if $s_{ij}(\theta)$ obeys an inequality analogous to (8) for arguments $z=(i,j)$ and $z'=(i',j')$, and for $\theta' \succeq \theta$.

Monotonicity is not needed for proportionately upcrossing synergy; we only require that positive synergy absolutely increase in θ more than negative synergy does.[12]

Proposition 4. *Assume (A2)* synergy is upcrossing in θ*, (B3)* synergy is one-crossing in types, and (C1) proportionately upcrossing synergy*. Sorting increases in θ in generic finite type models, or with continuum types if synergy strictly one-crosses in types.*

We prove that *Proposition 3 implies Proposition 4.* If synergy is upcrossing in θ and proportionately upcrossing, then summed rectangular synergy is upcrossing in θ, while marginal rectangular synergy is one-crossing in types.

As synergy is proportionately upcrossing if it is increasing in θ and monotone in types[13], *Proposition 4 implies Proposition 1,* finishing the logical result chain.

Appendix §B.2 derives a simple smooth condition for (8): synergy is proportionately upcrossing if it is *smoothly log-supermodular (LSPM)*, namely $\sigma = \phi_{12}$ obeys

$$\sigma_{ij}\sigma \geq \sigma_i \sigma_j \tag{9}$$

Corollary 1. *Assume a continuum of types, with synergy upcrossing in θ (A2), strictly one-crossing in types (B3), and smoothly LSPM (C2). Then sorting is increasing in θ.*

Figure 11 presents an example in which synergy is both upcrossing in θ and in types, but in which sorting *falls* in θ. To verify that synergy is not proportionately upcrossing in this figure, let $z=(2,1)$, $z'=(2,2)$, $t=\theta'$, and $t'=\theta''$. Then:

$$\phi_{12}^-(z \wedge z', t)\phi_{12}^+(z \vee z', t') = 2 \cdot 4 = 8 < 20 = 5 \cdot 4 = \phi_{12}^-(z, t')\phi_{12}^+(z', t)$$

[12] Assume negative synergy at couple z, and positive at a higher couple $z' = z \vee z' \succeq z \vee z' = z$. Then (8) says that $\phi_{12}^+(z', \theta')/\phi_{12}^+(z', \theta) \geq \phi_{12}^-(z, \theta')/\phi_{12}^-(z, \theta)$.

[13] $(z \vee z', \theta') \succeq (z', \theta) \Rightarrow \phi_{12}^+(z \vee z', \theta') \geq \phi_{12}^+(z', \theta)$, and $(z, \theta') \succeq (z \wedge z', \theta) \Rightarrow \phi_{12}^-(z \wedge z', \theta) \geq \phi_{12}^-(z, \theta')$.

	x_1	x_2	x_3
y_3	6	7	**11**
y_2	**4**	6	6
y_1	0	**4**	6

$\rightarrow$

	x_1	x_2	x_3
y_3	6	**13**	18
y_2	**4**	9	10
y_1	1	4	**10**

	x_1x_2	x_2x_3
y_2y_3	-1	4
y_1y_2	-2	-2

$\rightarrow$

	x_1x_2	x_2x_3
y_2y_3	2	4
y_1y_2	-1	-5

Figure 11: **Proportionately Upcrossing Failure.** Payoffs are at left, and synergies at right: The unique efficient matching falls from NAM3 to PAM2 as θ' shifts up to θ''. Synergy is upcrossing in θ and in types, but is not proportionately upcrossing, as Proposition 4 requires. Notice how sorting falls in θ.

This example violates both time-series and cross-sectional premises in Proposition 2. In particular, rectangular synergy is not upcrossing in types at θ'', since $2+(-1) > 0 > 4+(-5)$. And rectangular synergy is not upcrossing in θ, since $4+(-2) > 0 > 4+(-5)$. This latter failure is precisely why the optimal matching falls from NAM3 to PAM2.

6 Increasing Sorting and Type Distribution Shifts

Distributional shifts can be formally embedded in production functions, and thus allow us to use our comparative statics theory to deduce sorting predictions for changes in the type distributions $G(\cdot|\theta)$ and $H(\cdot|\theta)$. We say that X *types shift up (down)* in θ if $G(\cdot|\theta)$ stochastically increases (decreases) in θ, i.e. $G(\cdot|\theta') \leq G(\cdot|\theta)$ if $\theta' \succeq \theta$. Similarly, Y *types shift up (down)* in θ if $H(\cdot|\theta)$ stochastically increases (decreases) in θ.

The PQD order introduced in §3 only ranks matching distributions with the *same* marginals G and H. To overcome this, we consider sorting in quantile space. Label every type by its *quantile* in the distribution, so $p = G(X(p,\theta)|\theta)$ and $q = H(Y(q,\theta)|\theta)$. Next for any matching distribution consider the associated bivariate *copula* which defines the sorting by quantiles, namely $C(p,q) = M(X(p,\theta), Y(q|\theta))$. The copula is the matching distribution defined on quantiles (p,q) rather than types (x,y). We say that *quantile sorting* is higher at M'' than M' when the associated copulas are ranked $C'' \succeq_{PQD} C'$; namely, when C'' has more mass than C' in all lower and upper orthants in (p,q) space. The quantile sorting order generalizes the PQD order. For if M'' and M' share the same marginals, then $C'' \succeq_{PQD} C'$ if and only if $M'' \succeq_{PQD} M'$. And since all copulas have uniform marginals by definition, we can compare two copulas in the PQD order even if the associated matching distributions do not share marginals.

By Lemma 1, greater quantile sorting order reduces the average geometric distance between matched quantiles, and raises the covariance across matched quantile pairs, and the coefficient in linear regression of male on female match partner quantiles.

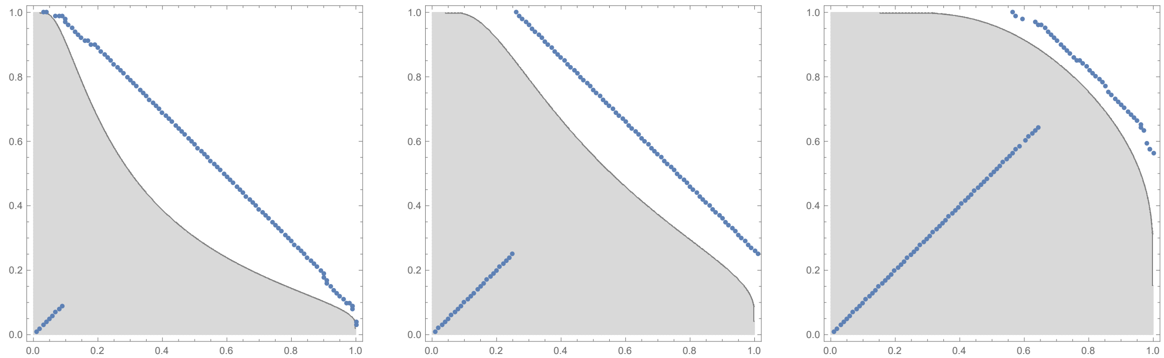

Figure 12: **Distribution Shift Example for Corollary 2.** These graphs depict optimally matched quantile pairs (blue dots) given an exponential distribution on types $G(x|\theta) = 1 - e^{-x/\theta}$ and $H(y|\theta) = 1 - e^{-y/\theta}$, and quadratic production $xy - (xy)^2$. By Corollary 2, *quantile sorting increases as θ falls*, since synergy is falling in types. The plots depict $\theta = 1, 2/3, 1/3$ at left, middle, and right.

Corollary 2. *Quantile sorting increases if types shift up (down):*
(a) generically with finite types, if synergy is non-decreasing (non-increasing) in types;
(b) given G and H absolutely continuous, if synergy is increasing (decreasing) in types.

For some insight into the proof in §C.5, consider the *quantile production function* $\varphi(p, q|\theta) \equiv \phi(X(p, \theta), Y(q, \theta))$ with *quantile synergy*:

$$\varphi_{12}(p, q|\theta) = \phi_{12}(X(p, \theta), Y(q, \theta))X_p(p, \theta)Y_q(q, \theta) \tag{10}$$

For concreteness, assume synergy ϕ is increasing in types, and that θ stochastically shifts up types. Then $\phi_{12}(X(p, \theta), Y(q, \theta))$ is increasing in quantiles p, q and θ. But we cannot conclude that *quantile synergy* is increasing in q and θ since (10) includes X_p and Y_q, which need not be monotone in q or θ. Given positive derivatives, quantile synergy is upcrossing in types and θ. We verify in §C.5 that the premise of Corollary 2 implies that of Proposition 3. Figure 12 depicts this result for quadratic production.

7 Economic Applications

7.1 Diminishing Returns

We wish to highlight a key property of the marriage model, that *diminishing returns reduces match synergies, and increasing returns amplifies them.* Firstly, note that a convex transformation of any SPM function is still SPM (Topkis, 1998). Put differently, concavity undermines SPM, and just as well, convexity undercuts SBM. Let's focus on the former: Assume that a type x worker on a type y machine has an increasing and

SPM output $q(x, y)$. Assume diminishing profit to output captured by the increasing and concave function ψ, profit is then $\phi(x, y|\theta) = \psi(q(x, y)|\theta)$. The synergy function

$$\phi_{12} = \psi'(q|\theta)\left[q_{12} + \frac{\psi''(q|\theta)}{\psi'(q|\theta)}q_1 q_2\right]$$

rises in complementarity q_{12} and falls in the Arrow-Pratt risk aversion measure $-\psi''/\psi'$.

To shed light on this tradeoff, assume $z = q(x, y) = xy$. We claim that sorting increases in θ if relative risk aversion $-z\psi''(z|\theta)/\psi'(z|\theta)$ falls in z and θ. That is, sorting falls as this concavity measure rises, provided it is falling in xy. We prove this in Appendix E by showing that synergy is smoothly LSPM when relative risk aversion is decreasing in z and θ. Thus, sorting increases in θ, by Proposition 4.

7.2 From Weakest to Strongest Link Technologies

Match partnerships differ by how differentially responsive output is to the types. One extreme case is the "weakest link" technology, namely, the SPM function $\min(x, y)$. Equally shared tasks, like jointly lifting a couch, have this flavor. Oppositely, the "strongest link" technology is the SBM function $\max(x, y)$. More generally, the lesser type matters more in a *weak link technology*. For a smooth technology ϕ, the rate of technical substitution therefore obeys $r(x, y) \equiv \phi_1(x, y)/\phi_2(x, y) \gtrless 1$ for $x \lessgtr y$. The opposite inequalities hold for a *strong link technology*, since the higher type matters more. Insurance exemplified this: the higher agent helps the lesser. We argue that *match synergies are higher with weak link, and lower with strong link, technologies.*

The CES technology $q(x, y) = (x^{-\rho} + y^{-\rho})^{-1/\rho}$ is a helpful tractable class. It is weak link and SPM when $\rho \geq -1$, and otherwise is strong link and SBM. The limits $\rho \to \pm\infty$ are the weakest and strongest link technologies, $\min(x, y)$ and $\max(x, y)$.

Let's integrate the lesson of §7.1. Assume diminishing returns to output q, namely, $\phi(x, y) = \alpha q(x, y) - \beta q(x, y)^2$, with $\alpha, \beta > 0$ and $\alpha > 2\beta q(1, 1)$. Synergy is continuous in ρ, and when $\rho = -1$, synergy is $-2\beta < 0$. Appendix E shows that synergy is upcrossing in ρ and strictly positive for ρ sufficiently large. All told, there exist $\bar{\rho} > \underline{\rho} > -1$ such that production is SBM (yielding NAM) for all $\rho < \underline{\rho}$ and SPM (giving PAM) for $\rho > \bar{\rho}$. We apply Proposition 3 to show that *sorting is increasing in ρ, for all $\rho \in [0, \overline{\rho}]$*.

A famous special case of strong link technologies arises with optimal ex post role assignment. Kremer and Maskin (1996) assume that agents are assigned either to the manager or deputy roles, where $x^\theta y^{1-\theta}$ is output if x is the manager and y the deputy, and $\theta \in [0, 1/2]$. As a unisex model, match output is then the maximum of two SPM functions $\max\{x^\theta y^{1-\theta}, x^{1-\theta} y^\theta\}$ — but is neither SPM nor SBM, since minimization preserves SPM, and maximization preserves SBM.

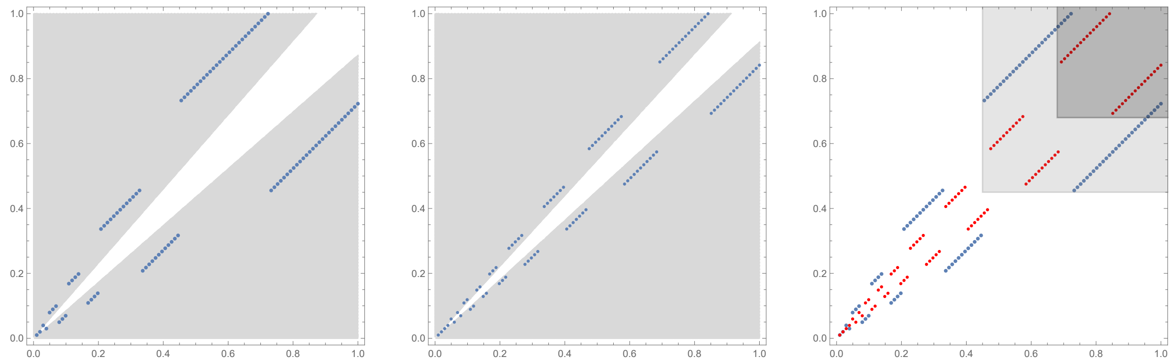

Figure 13: **Kremer-Maskin Synergies and Matching.** These graphs depict optimal matchings for production (11) with $\varrho = -20$ and a uniform distribution on 100 types. In the left graph $\theta = 0.4$ and rises to $\theta = 0.45$ in the middle. Synergy is positive on the shaded region, and is not one-crossing in types. *So our sorting monotonicity theory is silent here.* Indeed, the matching for $\theta = 0.45$ has more (red) couples in the dark rectangle in the right graph, while the matching for $\theta = 0.4$ has more (blue) couples in the light rectangle. Appendix E proves sorting is nowhere decreasing in θ.

To apply our theory, we introduce indexed smooth production functions converging to it as $\varrho \to -\infty$:

$$\phi(x, y|\theta, \rho) = x^\theta y^\theta \left(x^{-\rho} + y^{-\rho}\right)^{\frac{2\theta-1}{\rho}} \tag{11}$$

The x, y cross partial of the smooth function $\phi(x, y|\theta, \rho)$ in (11) is $+, -, +$ as types increase (Figure 13). Thus, our essential assumption of Proposition 2 that rectangular synergy is one-crossing in types fails. Hence, sorting is not monotone in Figure 13. Yet while sorting is not monotone, it cannot PQD fall as (θ, ρ) increases. See Appendix E.

7.3 Moral Hazard with Endogenous Contracts

Serfes (2005) explores a pairwise matching principal-agent model. The output for any project is the sum of the agent's unobservable effort and a mean zero Gaussian error. Project variances $y \in [\underline{y}, \overline{y}]$ vary across principals, while agents differ by risk aversion parameter $x \in [\underline{x}, \overline{x}]$, and share a scalar dis-utility of effort $\theta > 0$. Contracts are signed after matching takes place; they specify the agent's wage as a function of realized output. Serfes derives (his equation (2)) the equilibrium expected output of an (x, y) match:

$$\phi(x, y|\theta) = \frac{1}{2\theta\,(1 + \theta x y)} \quad \Rightarrow \quad \phi_{12}(x, y|\theta) = \frac{\theta x y - 1}{2\,(1 + \theta x y)^3} \tag{12}$$

Serfes observes that synergy is globally negative for $\theta \bar{x}\bar{y} < 1$ and globally positive for $\theta \underline{x}\underline{y} > 1$. Thus, by Becker's Sorting Result, NAM obtains for $\theta < (\bar{x}\bar{y})^{-1}$ and PAM obtains for $\theta > (\underline{x}\underline{y})^{-1}$. This result reflects two counterveiling forces for sorting.

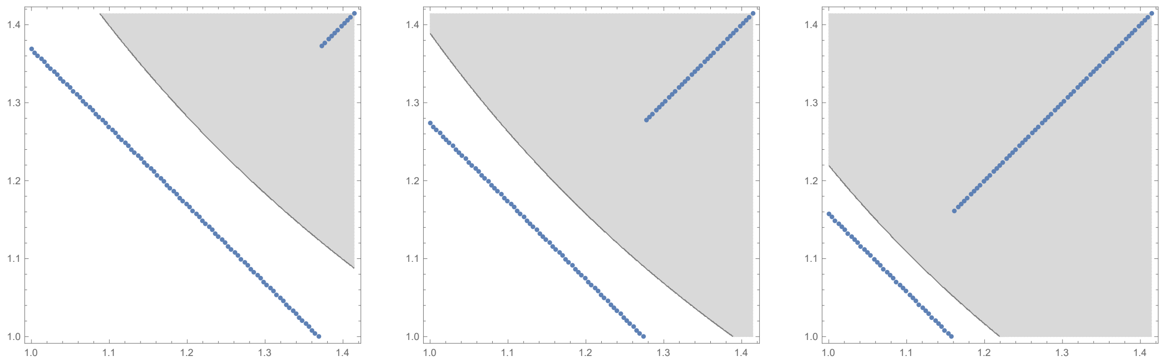

Figure 14: **Increasing Sorting in the Principal-Agent Model.** These graphs depict optimal matched pairs (blue dots) for a uniform distribution on 100 types of principals and agents. Sorting rises from left to right as θ rises on $\{0.65, 0.72, 0.82\}$.

First, if all contracts were the same, then efficient insurance across principal-agent pairs favors NAM: less risk averse agents work on higher variance projects. But the slope of the equilibrium wage contract is $(1+\theta xy)^{-1}$; and thus, the incentives to provide effort are SPM for high types. The sign of synergy (12) implies that the insurance effect dominates for low types, and the incentive effect dominates for high types.[14]

Our theory addresses some of the cases where Serfes is silent, when $\theta\bar{x}\bar{y} > 1 \geq \theta\underline{xy}$. We claim: *Sorting is increasing in θ when $\bar{x}\bar{y} \leq 2\underline{xy}$ (‡).* To see this, assume $\theta' > \theta$. If $\theta\bar{x}\bar{y} < 1$, then synergy (12) is globally negative at θ, and so NAM uniquely optimal. If $\theta'\underline{xy} > 1$, then synergy is globally positive at θ', and so PAM uniquely optimal. In both cases, sorting is weakly higher at θ' than θ. Now assume $\theta'\underline{xy} \leq 1 < \theta\bar{x}\bar{y}$. Then $\theta'\bar{x}\bar{y} \leq 2\theta'\underline{xy} \leq 2$ by (‡) and $\theta'\underline{xy} \leq 1$. Thus, $\theta xy < \theta' xy \leq 2$ for all (x, y), and so synergy (12) is increasing in θxy — for $(t-1)/(1+t)^3$ is increasing for $t \in (0, 2]$. Sorting increases in θ by Proposition 1, as in Figure 14. Since synergy increases in types when PAM is suboptimal, quantile sorting increases when types shift up, by Corollary 2.

The big picture is that the higher is the disutility of effort θ, the greater are the incentive difficulties of matching, as reflected in the lower slope of the wage contract.

7.4 Mentor-Protégé Learning Dynamics

Dynamic matching with evolving types can be understood through the lens of match synergies. Let's assume a two period model, with pairwise matching in periods one

[14] Ackerberg and Botticini (2002) investigate matching between landowners (principals) and tenants (agents) in 15th century Tuscany. Matched crop-tenant pairs exhibit positive covariance in crop types (project variance y) and tenant wealth (risk aversion x). But since match sorting is imperfect (not PAM), our theory provides a framework for analyzing changes in crop-tenant matching across markets.

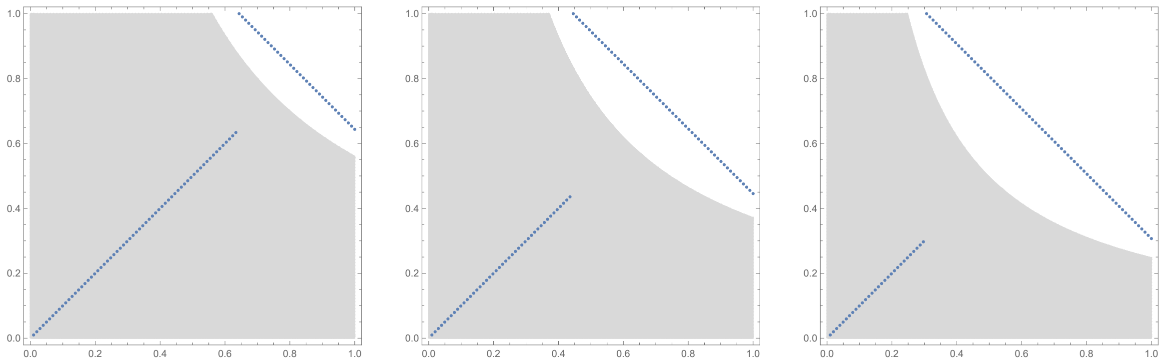

Figure 15: **Increasing Sorting with Peer Learning.** These graphs depict optimally matched pairs with static output $\phi^0(x,y) = \sqrt{xy}$ and transitions $\tau = x + 0.7(y-x) + 0.5(x^2 - xy)$ and a uniform distribution on 100 types. Sorting falls as the discount factor rises from $\delta = 0.4$ (left) to $\delta = 0.45$ (middle) to $\delta = 0.5$ (right).

and two. Let $\phi^0(x,y)$ be the increasing and SPM match output of types x and y.

We capture learning dynamics by the increasing *transition function* τ. Specifically, after producing output in period one, types x and y evolve to new types $x' = \tau(x,y)$ and $y' = \tau(y,x)$ in period two. For matching between workers within a firm, τ describes learning from co-workers. In a neighborhood sorting application, τ may reflect peer influences on children. Or in a procreation context, couple (x,y) produces offspring of type $\tau(x,y)$. In this latter case, $\tau(x,y) = \max(x,y)$ and $\tau = \min(x,y)$ formalize the respective extremes of dominant and recessive type transmission — namely, one or both high achieving parent suffices for high achieving children.

Matching must be statically optimal in period two, and thus PAM occurs.[15] For instance, in the partnership model, the social planner has period one payoff:

$$\phi(x,y) = (1-\delta)\phi^0(x,y) + \tfrac{\delta}{2}\left[\phi^0(\tau(x,y),\tau(x,y)) + \phi^0(\tau(y,x),\tau(y,x))\right]$$

given discount factor δ. So synergy ϕ_{12} is a $(1-\delta,\delta)$ weighted average of static synergy $\phi^0_{12} > 0$ and *dynamic synergy* — namely, if τ is twice differentiable, the first term is

$$\left[\phi^0(\tau(x,y),\tau(x,y))\right]_{12} = \left(\phi^0_{11} + 2\phi^0_{12} + \phi^0_{22}\right)\tau_1\tau_2 + \left(\phi^0_1 + \phi^0_2\right)\tau_{12} \tag{13}$$

Since τ is increasing, the first term in (13) is positive when $\phi^0(x,x)$ is convex, but

[15] Anderson and Smith (2010) consider an infinite horizon with stochastic type transitions. In a special case of the model where types are the common knowledge chance that an agent is high (vs. low) productivity, they show that synergy is negative for (x,y) close to $(0,0)$ or $(1,1)$ with sufficient patience. Thus, PAM cannot be optimal given sufficient patience.

negative when $\phi^0(x,x)$ is concave. That is, convexity pushes toward positive synergy and concavity toward negative synergy, as in §7.1. But in this evolving type world, negative synergy may also reflect a submodular transition function τ. This arises in learning environments, where the lower type learns from the higher, as a protege from a mentor. In particular, given the normalization $\tau(x,x) = x$, strictly SBM τ implies:

$$\tau(x,y) + \tau(y,x) > \tau(x,x) + \tau(y,y) \quad \Leftrightarrow \quad \tau(x,y) - x > y - \tau(y,x)$$

So when unequal types match, the higher partner pulls up the lower more than the latter pulls him down — as in a workplace when skilled co-workers pass on insights.

In particular, Herkenhoff, Lise, Menzio, and Phillips (2018) find negative dynamic synergy in such a setting.[16] Our model affords comparative statics in the discount factor. Since synergy is increasing in $1-\delta$, the *time-series premise* of each of our increasing sorting results is met. But, stronger assumptions are required for the *cross-sectional* assumptions. The most transparent case is when static synergy and dynamic synergy (13) are both monotone in types in the same direction. Then sorting falls in δ, by Proposition 1. Figure 15 shows this comparative static in a parametric example.

8 Conclusion

Becker's finding that complementarity (or supermodularity) yields positive sorting launched the immense literature on pairwise matching. But an impassable wall of mathematical complexity has prevented any general theory for non-assortative matching — despite many economic models needing such a theory. This paper bypasses the solution of the optimal matching, and nevertheless derives the missing general theory for comparative statics. We argue that the PQD stochastic order captures the economic notion of increasing sorting, and then answer the typical comparative static economist want: what productivity or type distribution shifts increase sorting in the PQD order?

We show that sorting increases if synergy globally increases and synergy crosses everywhere from negative to positive as types rise, and obey a cross-sectional assumption — e.g. it changes sign at most once on rectangles as the rectangle shifts northeast, or if a purely local condition called proportionate upcrossing is met.

We revisit the matching literature since 1990, quickly deriving and strengthening their findings, using our theory. Our paper offers a tractable foundation for future theoretical and empirical analysis of matching. A subtle and valuable direction for future work is a multidimensional extension of our theory (Lindenlaub, 2017).

[16] They estimate a matching model with search frictions and find SPM static production, but negative dynamic synergy. Synergy is positive for low types and negative synergy for high types.

We assumed an equal mass of men and women, like Becker. If types are imagined as quality, this is WLOG: lowest men are queued out if men are in surplus. Extending our increasing sorting results to a horizontal model of variety types is an open question.

We considered the planner's sorting exercise, and are silent on transfers. Future research could characterize the behavior of wage changes as sorting increases.

A Match Output Reformulation: Derivation of (4)

Proof: Summing $\sum_{i=1}^n \sum_{j=1}^n f_{ij} m_{ij}$ by parts in j and then i yields:

$$\begin{aligned}
\textstyle\sum_{i=1}^n \left[\sum_{j=1}^n f_{ij} m_{ij}\right] &= \sum_{i=1}^n \left[f_{in} \sum_{j=1}^n m_{ij} - \sum_{j=1}^{n-1} [f_{i,j+1} - f_{ij}] \sum_{k=1}^j m_{ik} \right] \\
&= \sum_{i=1}^n f_{in} - \sum_{j=1}^{n-1} \sum_{i=1}^n [f_{i,j+1} - f_{ij}] \sum_{k=1}^j m_{ik} \\
&= \sum_{i=1}^n f_{in} - \sum_{j=1}^{n-1} \left([f_{n,j+1} - f_{n,j}] \sum_{\ell=1}^n \sum_{k=1}^j m_{\ell k} - \sum_{i=1}^{n-1} s_{ij} \sum_{\ell=1}^i \sum_{k=1}^j m_{\ell k} \right) \\
&= \sum_{i=1}^n f_{in} - \sum_{j=1}^{n-1} \left([f_{n,j+1} - f_{nj}]\, j - \sum_{i=1}^{n-1} s_{ij} M_{ij} \right)
\end{aligned}$$

B Integral Preservation of Upcrossing Properties

B.1 Integral Preservation of Upcrossing Functions on Lattices

Given a real or integer lattice[17] $Z \subseteq \mathbb{R}^N$ and poset $(\mathcal{T}, \succeq)$, the function $\sigma : Z \times \mathcal{T} \to \mathbb{R}$ is proportionately upcrossing[18] if $\forall z, z' \in Z$ and $t' \succeq t$.[19]

$$\sigma^-(z \wedge z', t)\sigma^+(z \vee z', t') \geq \sigma^-(z, t')\sigma^+(z', t) \tag{14}$$

Theorem 1. *Let $\sigma(z,t)$ be proportionately upcrossing. Then $\Sigma(t) \equiv \int_Z \sigma(z,t) d\lambda(z)$ is weakly upcrossing in t, and upcrossing in t if $\sigma(z,t)$ is upcrossing in t.*

[17]We prove a stronger than needed result, as it applies to general lattices; we just need it for $\mathbb{R}^2$.

[18]Proportionately upcrossing implies *weakly upcrossing*; namely, $\sigma(z,t) > 0$ implies $\sigma(z', t') \geq 0$ for all $(z', t') \succeq (z, t)$. To see this, fix $t = t'$ and suppress t. If $z' \succeq z$, inequality (14) is an identity. If $z \succ z'$, inequality (14) becomes $\sigma^-(z')\sigma^+(z) \geq \sigma^-(z)\sigma^+(z')$, which precludes $\sigma(z) < 0 < \sigma(z')$.

[19]This result is related to Theorem 2 in Quah and Strulovici (2012). They do not assume (14). Rather, they assume σ is upcrossing in (z, θ), and a time a series condition: signed ratio monotonicity. Our results are independent, but overlap more closely for our smoothly LSMP condition in §B.2.

This generalizes an information economics result by Karlin and Rubin (1956): *If $\sigma_0(z)$ is upcrossing in $z \in \mathbb{R}$, and $\sigma_1 \geq 0$ is LSPM, then $\int \sigma_0(z)\sigma_1(z,t)d\lambda(z)$ is upcrossing.* Our result subsumes theirs when $n = 1$ and $\sigma = \sigma_0\sigma_1$ is proportional upcrossing.

Proof: Karlin and Rinott (1980) prove the following: *If functions $\xi_1, \xi_2, \xi_3, \xi_4 \geq 0$ obey $\xi_3(z \vee z')\xi_4(z \wedge z') \geq \xi_1(z)\xi_2(z')$ for $z \in Z \subseteq R^N$, then for all positive measures λ:*[20]

$$\int \xi_3(z)d\lambda(z) \int \xi_4(z)d\lambda(z) \geq \int \xi_1(z)d\lambda(z) \int \xi_2(z)d\lambda(z) \tag{15}$$

Now, if $t' \succeq t$, then (14) reduces to $\xi_3(z \vee z')\xi_4(z \wedge z') \geq \xi_1(z)\xi_2(z')$ for the functions:

$$\xi_1(z) \equiv \sigma^+(z,t), \quad \xi_2(z) \equiv \sigma^-(z,t'), \quad \xi_3(z) \equiv \sigma^+(z,t'), \quad \xi_4(z) \equiv \sigma^-(z,t)$$

Thus, by (15):

$$\int \sigma^+(z,t')d\lambda(z) \int \sigma^-(z,t)d\lambda(z) \geq \int \sigma^+(z,t)d\lambda(z) \int \sigma^-(z,t')d\lambda(z) \tag{16}$$

This precludes $\int \sigma^+(z,t)d\lambda(z) > \int \sigma^-(z,t)d\lambda(z)$ and $\int \sigma^+(z,t')d\lambda(z) < \int \sigma^-(z,t')d\lambda(z)$, simultaneously. And thus, $\Sigma(t) > 0$ implies $\Sigma(t') \geq 0$, proving weakly upcrossing.

We now argue Σ upcrossing. First assume $\Sigma(t) > 0$. Then $\int \sigma^+(z,t)d\lambda(z) > \int \sigma^-(z,t)d\lambda(z)$. By (16), either $\int \sigma^+(z,t')d\lambda(z) > \int \sigma^-(z,t')d\lambda(z)$, or $\int \sigma^+(z,t')d\lambda(z) = \int \sigma^-(z,t')d\lambda(z) = 0$. But the latter is impossible, since $\int \sigma^+(z,t')d\lambda(z) = 0$ implies $\int \sigma^+(z,t)d\lambda(z) = 0$, as $\sigma(z,t)$ is upcrossing in t — contradicting $\Sigma(t) > 0$. So $\Sigma(t') > 0$.

Next, posit $\Sigma(t) = 0$, then $\int \sigma^+(z,t)d\lambda(z) = \int \sigma^-(z,t)d\lambda(z)$. By (16), either $\int \sigma^+(z,t')d\lambda(z) \geq \int \sigma^-(z,t')d\lambda(z)$, and so $\Sigma(t') \geq 0$. Or, we have $\int \sigma^+(z,t)d\lambda(z) = \int \sigma^-(z,t)d\lambda(z) = 0$, whereupon $\int \sigma^-(z,t')d\lambda(z) = 0$ — as $\sigma(z,t)$ is upcrossing in t, and so $\sigma^-(z,t)$ is downcrossing. Thus, $\int \sigma^+(z,t')d\lambda(z) \geq \int \sigma^-(z,t')d\lambda(z)$, or $\Sigma(t') \geq 0$. $\square$

B.2 Proportionately Upcrossing and Log-supermodularity

We now introduce a sufficient condition for (14) that emphasizes the link between log-complementarity and proportional upcrossing. Let $\theta \in \mathbb{R}$, and call $\sigma(z,\theta)$ *smoothly signed log-supermodular (LSPM)* if its derivatives obey the inequality $\sigma_{ij}\sigma \geq \sigma_i\sigma_j$.

Theorem 2. *If $\sigma(z,\theta)$ is upcrossing and smoothly signed LSPM, then σ obeys* (14).

STEP 1: RATIO ORDERING. Abbreviate $w = (z,\theta) \in \mathbb{R}^{N+1}$. Assume $\hat{w} \geq w$, sharing

[20] The proof for the integer lattice requires that λ be a counting measure. Also true: if λ does not place all mass on zeros of σ, then $\Sigma(t) \equiv \int_Z \sigma(z,t)d\lambda(z)$ is upcrossing in t.

the i coordinate $w_i = \hat{w}_i$, with $\sigma(\bar{x}, w_{-i}) < 0 < \sigma(\hat{w})$ for some $\bar{x} > w_i$. Then we prove:

$$\sigma_i(x, w_{-i})\sigma(x, \hat{w}_{-i}) \geq \sigma_i(x, \hat{w}_{-i})\sigma(x, w_{-i}) \quad \forall\, x \in [w_i, \bar{x}] \tag{17}$$

Since σ is upcrossing, $\sigma(x, w_{-i}) < 0 < \sigma(x, \hat{w}_{-i})$ for all $x \in [w_i, \bar{x}]$. If (17) fails, then for some $x' \in [w_i, \bar{x}]$:

$$\frac{\sigma_i(x', w_{-i})}{\sigma(x', w_{-i})} > \frac{\sigma_i(x', \hat{w}_{-i})}{\sigma(x', \hat{w}_{-i})}$$

This contradicts smoothly LSPM, as $(\sigma_i/\sigma)_j \geq 0$ for all $\sigma \neq 0$ and $i \neq j$. So (17) holds. Given $\sigma(x, \hat{w}_{-i}) \neq 0$, the ratio $\sigma(x, w_{-i})/\sigma(x, \hat{w}_{-i})$ is non-decreasing in x on $[w_i, \bar{x}]$, so that:

$$\frac{\sigma(w)}{\sigma(\hat{w})} \leq \frac{\sigma(\bar{x}, w_{-i})}{\sigma(\bar{x}, \hat{w}_{-i})} \tag{18}$$

Step 2: σ obeys (14). By assumption $\theta' \geq \theta$ (now a real). So if $(z, \theta') \leq (z \wedge z', \theta)$, we have $z \leq z'$ and $\theta' = \theta$, in which case (14) is an identity. If not $(z, \theta') \leq (z \wedge z', \theta)$, then let $i_1 < \cdots < i_K$ be the indices with $(z, \theta')_{i_k} > (z \wedge z', \theta)_{i_k}$ for $k = 1, \ldots, K$. Let's change $w^0 \equiv (z \wedge z', \theta)$ into $w^K \equiv (z, \theta')$ in K steps, $w^0, \ldots, w^K$, one coordinate at a time, and likewise $\hat{w}^0 \equiv (z', \theta)$ into $\hat{w}^K \equiv (z \vee z', \theta')$, changing coordinates in the same order. Notice that $w^{k-1}_{i_k} = \hat{w}^{k-1}_{i_k} = (z', \theta)_{i_k} < (z, \theta')_{i_k}$ and $\hat{w}^k \geq w^k$ for all k.

Now, inequality (14) holds if its RHS vanishes. Assume instead the RHS of (14) is positive for some $\theta' \geq \theta$, so that $\sigma(z, \theta') < 0 < \sigma(z', \theta)$; and so, replacing $\hat{w}^0 = (z', \theta)$ and $w^K = (z, \theta')$, we get $\sigma(w^K) < 0 < \sigma(\hat{w}^0)$. But then since the sequences $\{w^k\}$ and $\{\hat{w}^k\}$ are increasing and σ is upcrossing, we have $\sigma(w^k) < 0 < \sigma(\hat{w}^{k-1})$ for all k. Altogether, we may repeatedly apply inequality (18) to get:

$$\frac{\sigma(z \wedge z', \theta)}{\sigma(z', \theta)} \equiv \frac{\sigma(w^0)}{\sigma(\hat{w}^0)} \leq \frac{\sigma(w^k)}{\sigma(\hat{w}^k)} \leq \cdots \leq \frac{\sigma(w^K)}{\sigma(\hat{w}^K)} \equiv \frac{\sigma(z, \theta')}{\sigma(z \vee z', \theta')}$$

So given $\sigma(z \wedge z', \theta), \sigma(z, \theta') < 0 < \sigma(z', \theta), \sigma(z \vee z', \theta')$, inequality (14) follows from:

$$\frac{\sigma^-(z \wedge z', \theta)}{\sigma^+(z', \theta)} \geq \frac{\sigma^-(z, \theta')}{\sigma^+(z \vee z', \theta')} \qquad \square$$

C Omitted Proofs

C.1 Proof of Proposition 2: Increasing Sorting for Finite Types

Lemma 2. *An optimal matching is generically unique and pure for finite types.*

Proof: The optimal matching is generically unique, by Koopmans and Beckmann

(1957). A non-pure matching M is a mixture $M = \sum_{\ell=1}^{L} \lambda_\ell M_k$ over $L \leq n+1$ pure matchings $M_1, \ldots, M_n$, with $\lambda_\ell > 0$ and $\sum_\ell \lambda_\ell = 1$.[21] As the objective function (3) is linear, if the non-pure matching M is optimal, so is each pure matching M_ℓ. □

A. Big Picture of the Proof. We show, for all n, that matching models in some domain $\hat{\mathcal{D}}_n$ obey our sorting conclusion. Our induction proof argues the stronger claim that it holds on a larger recursively convenient domain $\mathcal{D}_n^* \supset \hat{\mathcal{D}}_n$.

B. Building Blocks

(a) Consider the generic case with unique optimal pure matchings μ, described by men partners $(\mu_1, \ldots, \mu_n)$ of women, or women partners $\omega = (\omega_1, \ldots, \omega_n)$ of men.

(b) To emphasize the dependence on the number of types n, write rectangular synergy as $S^n(r|\theta)$, and the *summed rectangular synergy* as $\mathcal{S}^n(K|\theta) = \sum_{k=1} S^n(r_k|\theta)$ for any finite set of non-overlapping rectangles $K = \{r_k\}$.

(c) We consider the *summed rectangular synergy dyad* $(\mathcal{S}^n(K|\theta'), \mathcal{S}^n(K|\theta''))$ for generic $\theta'' \succeq \theta'$. Let domain $\mathcal{D}_n$ be the space of summed rectangular synergy dyads $(\mathcal{S}^n(K|\theta'), \mathcal{S}^n(K|\theta''))$ that are each upcrossing in K on rectangles $\mathcal{R}$ and upcrossing in θ on $\{\theta', \theta''\}$ for any $K \in \mathcal{R}$. The domain $\hat{\mathcal{D}}_n \subseteq \mathcal{D}_n$ further insists that they be upcrossing in θ for finite sets of non-overlapping rectangles K. Proposition 2 assumes that summed rectangular synergy dyads are in $\hat{\mathcal{D}}_n$ for all n.

(d) *Removing couple* (i,j) from an n-type market *induces rectangular synergy* S_{ij}^{n-1} among the remaining $n-1$ types, satisfying the natural formula:

$$S_{ij}^{n-1}(r|\theta) \equiv S^n(r + \mathcal{I}_{ij}(r)|\theta) \quad \text{for} \quad \mathcal{I}_{ij}(r) = (\mathbb{1}_{r_1 \geq i}, \mathbb{1}_{r_2 \geq j}, \mathbb{1}_{r_3 \geq i}, \mathbb{1}_{r_4 \geq j}) \tag{19}$$

where $\mathcal{I}_{ij}(r)$ increments by one the index of the women $i' \geq i$ and men $j' \geq j$, *where the type indices refer to the original model whenever removing types* henceforth.

(e) To avoid ambiguity when changing the number n of types, we denote by (i_n, j_n) the ith highest woman and the jth highest man. Now, consider the sequence models with $\kappa = n+k, n+k-1, \ldots, n$ types induced by removing couple (i'_κ, j'_κ) at θ' and (i''_κ, j''_κ) at θ'' from the κ type model. We say the sequence of couples has *higher partners at* θ' *than* θ'' if $(i'_\kappa, j'_\kappa) \geq (i''_\kappa, j''_\kappa)$ and $i'_\kappa = i''_\kappa$ or $j'_\kappa = j''_\kappa$.

(f) Domain $\mathcal{D}_n^*$ is the set of summed rectangular synergy dyads $(\mathcal{S}^n(K|\theta'), \mathcal{S}^n(K|\theta''))$ induced by sequentially removing k optimally matched couples with higher partners at θ' than θ'' from dyads $(\mathcal{S}^{n+k}(K|\theta'), \mathcal{S}^{n+k}(K|\theta'')) \in \hat{\mathcal{D}}_{n+k}$, for some $k \in \{0, 1, \ldots\}$. So our earlier Figure 8 violated this property, removing a higher couple at θ'' than θ'.

C. Key Properties of our Domains and Pure Matchings.

[21] This follows from Carathéodory's Theorem. It says that non-empty convex compact subset $\mathcal{X} \subset \mathbb{R}^n$ are weighted averages of extreme points of $\mathcal{X}$. The extreme points here are the pure matchings.

Fact 1. *Fix a summed rectangular synergy dyad in* $\mathcal{D}^*_{n+1}$*. Removing couple* (i',j') *at* θ' *and* (i'',j'') *at* θ'' *induces such a dyad in* $\mathcal{D}^*_n$ *if* $(i',j')\geq(i'',j'')$ *and* $i'=i''$ *or* $j'=j''$*.*

Fact 2. *Given a summed rectangular synergy dyad in* $\mathcal{D}_{n+1}$*, removing couple* (i',j') *at* θ' *and* (i'',j'') *at* θ'' *induces a summed rectangular synergy dyad in* $\mathcal{D}_n$ *if* $\langle i'=i''$ *and* $j'\geq j''\rangle$ *or* $\langle j'=j''$ *and* $i'\geq i''\rangle$*.*

Proof: We prove this for $i'=i''$ and $j'\geq j''$. For any θ, rectangular synergy $S^n_{ij}(r|\theta)$ is upcrossing in r, needing fewer inequalities. To see that summed rectangular synergy is upcrossing in θ on rectangular sets in $\mathbb{Z}^2_{n-1}$, assume $S^n_{ij'}(r|\theta')\geq(>)0$ for some r. Then

$$\begin{aligned} S^{n+1}(r+\mathcal{I}_{ij'}(r)|\theta')\geq(>)0 &\Rightarrow S^{n+1}(r+\mathcal{I}_{ij''}(r)|\theta')\geq(>)\,0 \\ &\Rightarrow S^{n+1}(r+\mathcal{I}_{ij''}(r)|\theta'')\geq(>)\,0\Rightarrow S^n_{ij''}(r|\theta'')\geq(>)\,0 \end{aligned}$$

respectively, as (i) $S^{n+1}(r|\theta)$ is upcrossing for rectangles r, non-increasing $\mathcal{I}_{ij}$ in j, and $j''\leq j'$, and (ii) $S^{n+1}(r|\theta)$ is upcrossing in θ for rectangles r, and (iii) by (19). □

Fact 3. *The domains are nested:* $\hat{\mathcal{D}}_n\subseteq\mathcal{D}^*_n\subseteq\mathcal{D}_n$*.*

Proof: Trivially, $\hat{\mathcal{D}}_n\subseteq\mathcal{D}^*_n$, since we may set $k=0$ in the definition of $\mathcal{D}^*_n$.

To get $\mathcal{D}^*_n\subseteq\mathcal{D}_n$, pick $(\mathcal{S}^n(K|\theta'),\mathcal{S}^n(K|\theta''))\in\mathcal{D}^*_n$. This dyad is induced by removing k optimally matched couples with higher partners at θ' than θ'' from a dyad $(\mathcal{S}^{n+k}(K|\theta'),\mathcal{S}^{n+k}(K|\theta''))\in\hat{\mathcal{D}}_{n+k}\subseteq\mathcal{D}_{n+k}$, where $k\geq0$. For $\ell=1,\ldots,k$, induce dyads $(\mathcal{S}^{n+k-\ell}(K|\theta'),\mathcal{S}^{n+k-\ell}(K|\theta''))$, sequentially removing optimally matched couples. So $(\mathcal{S}^{n+k-\ell}(K|\theta'),\mathcal{S}^{n+k-\ell}(K|\theta''))\in\mathcal{D}_{n+k-\ell}$ for $\ell=1,\ldots,k$, as removed couples are ordered, as Fact 2 needs. So $(\mathcal{S}^n(K|\theta'),\mathcal{S}^n(K|\theta''))\in\mathcal{D}_n$. □

Fact 4. *If* $M\neq\hat{M}$ *are pure n-type matchings,* $\hat{\mu}_i>\mu_i$ *at some* i *and* $\hat{\omega}_j>\omega_j$ *at some* j*.*

Proof: Since $M\neq\hat{M}$, there is a highest type man j matched with woman $\hat{\omega}_j>\omega_j$. Logically then, woman $i=\hat{\omega}_j$ is matched to a lower man under M, i.e. $j=\hat{\mu}_i>\mu_i$. □

Adding a couple (i_0,j_0) to a matching μ creates a new matching $\hat{\mu}$ with indices of women $i\geq i_0$ and men $j\geq j_0$ renamed $i+1$ and $j+1$, respectively. Equivalently, this means inserting a row i and column j into the matching matrix m — with all 0's except 1 at position (i,j) — and shifting later rows and columns up one.

Fact 5. *Adding respective couples* $(1,\hat{m})\leq(1,m)$*, or* $(\hat{w},1)\leq(w,1)$*, to the n-type matchings* $\hat{\mu}\succeq_{PQD}\mu$ *preserves the PQD order for the resulting* $n+1$ *type matchings.*

Proof: We just consider adding couples $(1,\hat{m})\leq(1,m)$, as the analysis for $(\hat{w},1)\leq(w,1)$ is similar. For pure matchings μ, let $C^\mu(i_0,j_0)$ count matches by women $i\leq i_0$ with men $j\leq j_0$, and so call $C^\mu(0,j)=C^\mu(i,0)=0$. So $\hat{\mu}\succeq_{PQD}\mu$ *iff* $C^{\hat{\mu}}\geq C^\mu$.

By adding a couple $(1,m)$, the new count is:

$$\mathcal{C}^{\mu}_{m}(i,j) \equiv C^{\mu}\left(i-1, j-\mathbb{1}_{j\geq m}\right) + \mathbb{1}_{j\geq m} \quad \text{for all } i,j \in \{1,2,\ldots,n+1\}$$

To prove the step, we must show that if $\hat{\mu} \succeq_{PQD} \mu$, then $\mathcal{C}^{\hat{\mu}}_{\hat{m}} \geq \mathcal{C}^{\mu}_{m}$ for all $\hat{m} \leq m$.

By assumption $\hat{\mu} \succeq_{PQD} \mu$ and thus, $C^{\hat{\mu}} \geq C^{\mu}$. So since $\hat{m} \leq m$:

$$.\mathcal{C}^{\hat{\mu}}_{\hat{m}}(i,j) - \mathcal{C}^{\mu}_{m}(i,j) = \begin{cases} C^{\hat{\mu}}(i-1,j) - C^{\mu}(i-1,j) & \geq \ 0 \quad \text{for } j < \hat{m} \\ C^{\hat{\mu}}(i-1,j-1) + 1 - C^{\mu}(i-1,j) & \geq \ 0 \quad \text{for } \hat{m} \leq j < m \\ C^{\hat{\mu}}(i-1,j-1) - C^{\mu}(i-1,j-1) & \geq \ 0 \quad \text{for } j \geq m \end{cases}$$

To understand the middle line, note that this match count can be written as

$$C^{\hat{\mu}}(i-1,j-1) - C^{\mu}(i-1,j-1) - \left[C^{\mu}(i-1,j) - C^{\mu}(i-1,j-1) - 1\right]$$

As $C^{\mu}(i-1,j) - C^{\mu}(i-1,j-1) \leq 1$, this is at least $C^{\hat{\mu}}(i-1,j-1) - C^{\mu}(i-1,j-1) \geq 0$. ☐

D. The Induction Proof: Detailed Steps. Let M'_n and M''_n be uniquely optimal n type matchings at θ' and θ''. Proposition 2 assumes summed rectangular synergy dyads in $\hat{\mathcal{D}}_n$. Until Step 8, we work on the larger domain $\mathcal{D}^*_n$.

Premise $\mathcal{P}_n$: *Summed rectangular synergy dyad is in* $\mathcal{D}^*_n \Rightarrow M''_n \succeq_{PQD} M'_n$.

Step 1. *Base Case* $\mathcal{P}_2$: *Summed rectangular synergy dyad is in* $\mathcal{D}^*_2 \Rightarrow M''_2 \succeq_{PQD} M'_2$.

Proof: If not, then NAM is uniquely optimal at θ'' and PAM at θ'. Since $\mathcal{D}^*_2 \subseteq \mathcal{D}_2$ by Fact 3, rectangular synergy is upcrossing in θ. This precludes negative rectangular synergy at θ'' (NAM) and positive rectangular synergy at θ' (PAM). ☐

- A *pair* refers to two *couples*, such as (i_1, j_1) and (i_2, j_2).
- A pair is a *PAM pair* if $(i_1, j_1) < (i_2, j_2)$, and a *NAM pair* if $i_1 < i_2$ and $j_1 > j_2$.

Step 2. *If the summed rectangular synergy dyad is in* $\mathcal{D}^*_{n+1}$, *then neither* M'_{n+1} *nor* M''_{n+1} *includes a subset of types that match according to NAM1.*

Proof: We prove the stronger conclusion that neither M'_{n+1} nor M''_{n+1} includes a matched NAM pair above a matched PAM pair. Indeed, by Fact 3, $\mathcal{D}^*_{n+1} \subseteq \mathcal{D}_{n+1}$. So $S^{n+1}(r|\theta)$ is upcrossing in rectangles r for θ' and θ''. Also, PAM (NAM) is optimal for a pair *iff* $S^{n+1}(r|\theta) \geq (\leq) 0$ on rectangle r. As the optimal matching is unique, $S^{n+1}(r|\theta) \neq 0$ for all optimally matched pairs. ☐

Steps 3–8 impose premises $\mathcal{P}_2, \ldots, \mathcal{P}_n$. When then supposed by contradiction that $\mathcal{P}_{n+1}$ is not satisfied. Equivalently, we suppose by contradiction:

(‡‡): *In a model with summed rectangular synergy dyads in $\mathcal{D}^*_{n+1}$, the generically uniquely optimal matchings at $\theta'' \succ \theta'$ are not ranked $\mu'' \succeq_{PQD} \mu'$ ($\omega'' \succeq_{PQD} \omega'$).*[22]

Our cross-sectional assumption rules out NAM1 for any three type subset of agents. Steps 3–7 show this restriction along with the inductive hypothesis and (‡‡) implies that the optimal matching for θ'' must be NAM for some subset of types $\{1, 2, \ldots, m\}$ and a multi-type generalization of NAM3 under θ' for this same subset of types that we call NAM*; namely, (m, m) matched and the remaining types $\{1, 2, \ldots, m-1\}$ matched according to NAM. Step 8 then applies the cross sectional and time series properties of the space $\mathcal{D}^*_{n+1}$ to rule out such NAM to NAM* transitions as θ rises.

Step 3. *At states θ' and θ'', the matchings obey $\mu''_1 = \mu'_1 + 1 \geq 2$ and $\omega''_1 = \omega'_1 + 1 \geq 2$.*

We establish the first relationship. Symmetric steps would prove the second.

Proof of $\mu''_1 > \mu'_1$: If not, then $\mu''_1 \leq \mu'_1$. In this case, remove couple $(1, \mu'_1)$ at θ', and couple $(1, \mu''_1)$ at θ''. The remaining matching is PQD higher at θ'', by Induction Premise $\mathcal{P}_n$ and Fact 1. By Fact 5, if we add back the optimally matched pairs $(1, \mu'_1)$ and $(1, \mu''_1)$, then the PQD ranking still holds with $n + 1$ types, given $\mu''_1 \leq \mu'_1$, namely $\mu'' \succeq_{PQD} \mu'$. This contradiction to (‡‡) proves that $\mu''_1 > \mu'_1$. □

Proof of $\mu''_1 < \mu'_1 + 2$. If not, then $\mu''_1 \geq \mu'_1 + 2$. By Fact 4, choose a woman $i > 1$ with $\mu''_i < \mu'_i$. Remove couples (i, μ'_i) at θ', and (i, μ''_i) at θ''. Since $\mu''_i < \mu'_i$, the resulting matching is PQD higher at θ'' than θ', by Fact 1 and Premise $\mathcal{P}_n$. In the resulting model, woman 1 is not matched to a higher man at θ'' than θ'. This is impossible if $\mu''_1 \geq \mu'_1 + 2$, as $\mu''_1 - \mu'_1$ falls by at most 1 when removing man μ_i at θ' and μ''_i at θ''. □

Step 4. *The couple (ω''_1, μ''_1) is matched at θ', namely, $\mu'_{\omega''_1} = \mu''_1$ and $\omega'_{\mu''_1} = \omega''_1$.*

In words: the man matched to the lowest woman under θ'' and the woman matched to the lowest man under θ'' must match together under θ'.

Proof of $\mu'_{\omega''_1} \geq \mu''_1$ and $\omega'_{\mu''_1} \geq \omega''_1$: We prove the first inequality. If not, then $\mu'_{\omega''_1} < \mu''_1$. As man $\mu'_1 = \mu''_1 - 1$ is matched at θ' by Step 3, $\mu'_{\omega''_1} < \mu''_1 - 1 = \mu'_1$. Removing couple $(\omega''_1, \mu'_{\omega''_1})$ at θ' and $(\omega''_1, 1)$ at θ'', induces and n type matching that is PQD higher at θ'', by $\mathcal{P}_n$ and Fact 1. Since man $\mu'_{\omega''_1}$ removed at θ' and man 1 removed at θ'' are below $\mu'_1 = \mu''_1 - 1$, the match count at $(1, \mu'_1 - 1)$ is unchanged at θ'' and θ'. By Step 3, this count is higher at θ' than θ'', contradicting the n type matching PQD higher at θ''. □

Proof of $\mu'_{\omega''_1} = \mu''_1$ and $\omega'_{\mu''_1} = \omega''_1$: Just one strict inequality is impossible, as it over-matches some type: $\omega'_{\mu''_1} > \omega''_1$ and $\mu'_{\omega''_1} = \mu''_1$ or $\omega'_{\mu''_1} = \omega''_1$ and $\mu'_{\omega''_1} > \mu''_1$. Next

[22] We cannot apply Theorem 4 to rule out $\mu' \succeq_{PQD} \mu''$, since the time-series premise of Theorem 4 is stronger than the time-series assumption in Proposition 2.

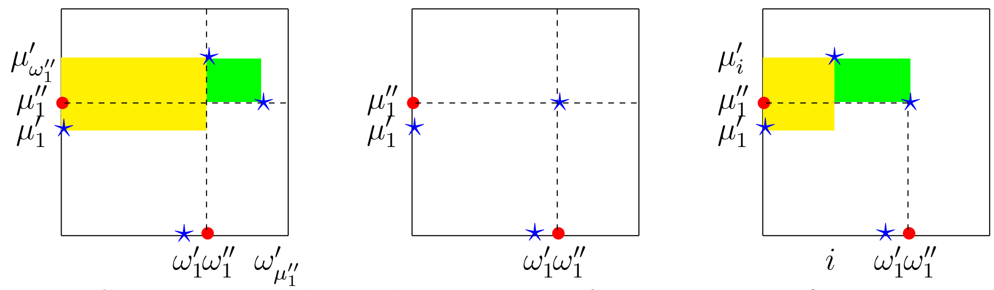

Figure 16: **Steps 3–5 in the Induction Proof.** In the counterfactual logic in Steps 3–5, stars and dots denote respective proposed matched pairs at θ' and θ''. Step 3 establishes that the index of partner for the lowest man (woman) under θ'' must be exactly one higher than the index for the lowest man (woman) under θ'. The left panel depicts the NAM pair (green) above the PAM pair (yellow) in Step 4. The middle panel depicts the conclusion of Step 4: man μ''_1 and woman ω''_1 must match under θ'. The right panel depicts the NAM pair above the PAM pair in Step 5-(a).

assume two strict inequalities. As $\mu'_{\omega''_1} > \mu''_1$, the θ' matching includes the PAM pair $(1, \mu'_1) < (\omega''_1, \mu'_{\omega''_1})$ — by Step 3 — and the higher NAM pair $(\omega''_1, \mu'_{\omega''_1})$ and $(\omega'_{\mu''_1}, \mu''_1)$. NAM pairs above PAM pairs violate Step 2 (left panel of Figure 16). □

The middle panel of Figure 16 depicts the takeout of Steps 3–4. We iteratively use this matching patter to show how (‡‡) greatly restricts the matching at θ' and θ''.

Step 5. $\mu'_1 \geq \mu'_i = \mu''_i - 1$ *for* $i = 1, \ldots, \omega'_1$ *and* $\omega'_1 \geq \omega'_j = \omega''_j - 1$ *for* $j = 1, \ldots, \mu'_1$.

Proof: We proved this for $i = 1$ and $j = 1$, and now prove the claimed ordering $\mu'_1 \geq \mu'_i = \mu''_i - 1$ for $i = 2, \ldots, \omega'_1$. By symmetry, $\omega'_1 \geq \omega'_j = \omega''_j - 1$ for $j = 2, \ldots, \omega'_1$.

Part (a): $\mu'_i < \mu'_1$ *for* $i = 2, \ldots, \omega'_1$. If not, then $\mu'_i \geq \mu'_1$ for some $2 \leq i \leq \omega'_1$. And since $\mu'_i = \mu'_1$ entails overmatching, we have $\mu'_i > \mu'_1$ for $i = 2, \ldots, \omega'_1$. Thus, μ' involves a PAM pair $(1, \mu'_1) < (i, \mu'_i)$. We claim that (i, μ'_i) and (ω''_1, μ''_1) constitutes a higher NAM pair, violating the upcrossing of $S(r|\theta)$ in r, by Step 2. Indeed, $i \leq \omega'_1 < \omega''_1$ (by the premise above, and Step 3, respectively). Also, $\mu'_i > \mu''_1$, since we have assumed $\mu'_i > \mu'_1$, and deduced $\mu'_1 = \mu''_1 - 1$ in Step 3, and, in Step 4, that μ''_1 is matched to ω''_1 at θ', and we just showed $\omega''_1 > i$. (See the right panel of Figure 16.) □

Part (b): $\mu'_i < \mu''_i$ *for* $i = 2, \ldots, \omega'_1$. If not, then $\mu'_i \geq \mu''_i$ for some $2 \leq i \leq \omega'_1$. Since $\mu'_i \geq \mu''_i$, if we remove couple (i, μ'_i) at θ' and couple (i, μ''_i) at θ'', then the resulting matching is PQD higher at θ'', by Fact 1 and $\mathcal{P}_n$. In the resulting matching, woman 1's partner is thus not higher at θ'' than θ'. But $\mu''_1 = \mu'_1 + 1$ by Step 3, and $\mu'_1 > \mu'_i \geq \mu''_i$ by part (a) and the premise of (b). Both removed men μ'_i and μ''_i are then strictly below μ'_1. So, woman 1's partner is still 1 higher at θ'' than θ'. Contradiction. □

Part (c): $\mu'_i \geq \mu''_i - 1$ *for* $i = 2, \ldots, \omega'_1$. If not, then $\mu'_{i^*} < \mu''_{i^*} - 1$ for some $2 \leq i^* \leq \omega'_1$.

Remove couple (ω_1'', μ_1'') at θ' (matched, by Step 4), and the couple $(\omega_1'', 1)$ at θ''. By Fact 1 and Assumption $\mathcal{P}_n$, the resulting matching is PQD higher at θ''.

But since $\omega_1'' > \omega_1'$ by Step 3, all women $i = 1, \ldots, \omega_1'$ remain. Each has a weakly lower partner at θ' than θ'', since we started with $\mu_i' < \mu_i''$ for $i = 1, \ldots, \omega_1'$ by Step 3 for $i = 1$, and part (b) for $i > 1$. Also, woman $i^* \leq \omega_1'$ has a strictly lower partner, as $\mu_{i^*}' < \mu_{i^*}'' - 1$. The resulting matching cannot be PQD higher at θ''. Contradiction. □

Step 6. *The matching μ'' is NAM among men and women at most $\omega_1'' = \mu_1'' \geq 2$.*

Proof of $\omega_1'' = \mu_1''$. By Steps 3 and 5, we get $\mu_1'' = \mu_1' + 1 \geq \mu_i''$ for $i = 1, \ldots, w_1' = \omega_1'' - 1$ and $\mu_1'' \geq 2 > 1 = \mu_{w_1''}''$. So in matching μ'', women $i \leq \omega_1''$ match with men $j \leq \mu_1''$. Hence, $\mu_1'' \geq \omega_1''$. Ditto, by Steps 3 and 5, $\omega_1'' \geq \omega_j''$ for $j = 1, \ldots, \mu_1''$, and in matching ω'', men $j \leq \mu_1''$ match with women $i \leq \omega_1''$. Hence, $\mu_1'' \leq \omega_1''$. Thus, $\mu_1'' = \omega_1'' \geq 2$. □

Proof of $\mu_i'' = \mu_1'' - i + 1$ for $1, \ldots, \omega_1''$. This is an identity at $i = 1$ and true at $i = \omega_1''$ by $\omega_1'' = \mu_1''$ (just proven) and $\mu_{\omega_1''}'' = 1$. So, henceforth assume $i \in \{2, \ldots, \omega_1'' - 1\}$. We claim that for all such i, $\mu_1' \geq \mu_i''$. Indeed, by Steps 3 and 5, $\mu_1'' = \mu_1' + 1 \geq \mu_i''$; and since we do not over match, $\mu_1'' \neq \mu_i''$ for $i \neq 1$. Since $\mu_1' \geq \mu_i''$, Step 5 yields equality $\omega_j' = \omega_j'' - 1$ at $j = \mu_i''$, and so $\omega_{\mu_i''}' = \omega_{\mu_i''}'' - 1 = i - 1$. But then since $\omega_{\mu_{i-1}'}' = i - 1$ and each woman has a unique partner, $\omega_{\mu_i''}' = i - 1$ implies $\mu_i'' = \mu_{i-1}'$. As $\mu_{i-1}' = \mu_{i-1}'' - 1$ by Step 5 and $i \leq \omega_1'' - 1 = \omega_1'$ (by our premise and Step 3), we have $\mu_i'' = \mu_{i-1}'' - 1$. □

An n-type pure matching μ is NAM* if $\mu_n = n$ and $\mu_i = n - i$ for $i = 1, \ldots, n - 1$, i.e. NAM among types $1, \ldots, n - 1$, so that NAM* = NAM3 when $n = 3$.

Step 7. *The matching μ' is NAM* among men and women at most $\omega_1'' = \mu_1'' \geq 2$.*

Proof: Steps 3, 5 and 6 imply $\mu_i' = \mu_i'' - 1 = \mu_1'' - i$ for $i = 1, \ldots, \omega_1' = \omega_1'' - 1$. Couple (ω_1'', μ_1'') matches under μ', by Step 4. So μ' is NAM* for types $1, \ldots, \mu_1'' = \omega_1''$. □

By Steps 6–7, μ'' is NAM and μ' is NAM* on types $1, \ldots, \omega_1'' = \mu_1'' \equiv k \geq 2$. Since NAM* $\succ_{PQD}$ NAM, if $k < n + 1$ then Premise $\mathcal{P}_k$ fails. Step 8 finishes the proof by showing that NAM at θ'' and NAM* at θ' is also impossible for $k = n + 1$ types.

NAM for men $\{i_1, \ldots, i_\ell\}$ and women $\{j_1, \ldots, i_\ell\}$ is $\{(i_1, j_\ell), (i_2, j_{\ell-1}), \ldots, (i_\ell, j_1)\}$. Rematching to NAM*, $\{(i_1, j_{\ell-1}), (i_2, j_{\ell-2}), \ldots, (i_\ell, j_\ell)\}$ changes payoffs by

$$\sum_{u=1}^{\ell-1} (f_{i_u, j_{\ell-u}} - f_{i_u, j_{\ell+1-u}}) + f_{i_\ell, j_\ell} - f_{i_\ell, 1} = \sum_{u=1}^{\ell-1} \left[(f_{i_\ell, j_{\ell+1-u}} - f_{i_\ell, j_{\ell-u}}) - (f_{i_u, j_{\ell+1-u}} - f_{i_u, j_{\ell-u}}) \right]$$

So *the payoff of NAM* less that of NAM on any subset of ℓ types equals* (suppressing the superscript on S)

$$\sum_{u=1}^{\ell-1} S(i_u, j_{\ell-u}, i_\ell, j_{\ell+1-u}) \tag{20}$$

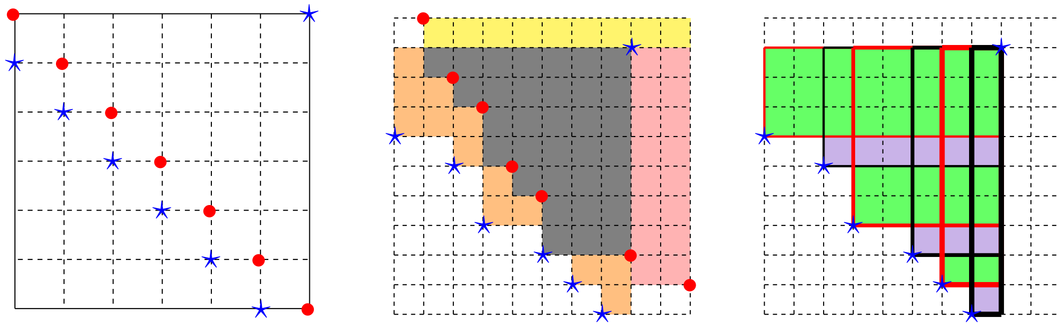

Figure 17: **Step 8 of Induction Proof.** At left, we rule out NAM for θ'' (dots) and NAM* for θ' (stars) with $n+1$ types. Adding $k-1$ couples weakly higher at θ' than θ'' produces the matches in the middle panel. Let K^G, K^O, K^P, K^Y be the grey, orange, pink, and yellow regions. By (20), the NAM* minus NAM difference is $\mathcal{S}^{n+k}(K^G \cup K^O|\theta') > 0$, as NAM* is optimal for θ'. But $\mathcal{S}^{n+k}(K^O|\theta') < 0$, as K^O is the union of rectangles, each below a NAM pair for θ''. So $\mathcal{S}^{n+k}(K^G|\theta') > 0$. By (20), the NAM* minus NAM difference is $\mathcal{S}^{n+k}(K^G \cup K^P \cup K^Y|\theta'') < 0$, negative by NAM optimal for θ''. Finally, $\mathcal{S}^{n+k}(K^Y|\theta'), \mathcal{S}^{n+k}(K^P|\theta') > 0$, as yellow and pink rectangles lie above a PAM pair for θ'. So $\mathcal{S}^{n+k}(K^G|\theta'') < 0$. But as $\mathcal{S}^{n+k}(K^G|\theta') > 0$, this contradicts summed rectangular synergy upcrossing in θ. At right, we depict Step 8(c).

Step 8. *NAM at θ'' $\Rightarrow\!\sim$ NAM* at θ' for summed rectangular synergy dyads in $\mathcal{D}^*_{n+1}$.*

Part (a): Contradiction Assumption. For $n+1$ types, posit NAM* and NAM uniquely optimal at θ' and θ'' (Figure 17, left panel). Induce summed rectangular synergy dyads in $\mathcal{D}^*_{n+1}$ by removing $k-1 \geq 0$ optimally matched couples with higher partners at θ' than θ'' (our earlier building block (f)) from a summed rectangular synergy dyad $(\mathcal{S}^{n+k}(K|\theta'), \mathcal{S}^{n+k}(K|\theta'')) \in \hat{\mathcal{D}}_{n+k}$. The θ' matching here is NAM* for men $\boldsymbol{i}' = (i'_1, \ldots, i'_{n+1})$ and women $\boldsymbol{j}' = (j'_1, \ldots, j'_{n+1})$, while the θ'' matching with these $n+k$ types is NAM for men $\boldsymbol{i}'' = (i''_1, \ldots, i''_{n+1})$ and women $\boldsymbol{j}'' = (j''_1, \ldots, j''_{n+1})$, with $(\boldsymbol{i}', \boldsymbol{j}') \leq (\boldsymbol{i}'', \boldsymbol{j}'')$ (Figure 17, middle panel).

Part (b): couple sets U', U'' with $\mathcal{S}^{n+k}(U''|\theta'') < 0 < \mathcal{S}^{n+k}(U'|\theta')$. For rectangles $r'_u \equiv (i'_u, j'_{n+1-u}, i'_{n+1}, j'_{n+2-u})$ and $r''_u \equiv (i''_u, j''_{n+1-u}, i''_{n+1}, j''_{n+2-u})$ define "upper sets":

- $U' \equiv \cup_{u=1}^{n} r'_u$, the union of the grey and orange rectangles in panel 2 of Figure 17
- $U'' \equiv \cup_{u=1}^{n} r''_u$, the union of the grey, yellow, and pink regions

As NAM* is uniquely optimal for the subsets of men $\boldsymbol{i}'$ and women $\boldsymbol{j}'$ at θ', it payoff-dominates NAM. Given linearity of summed rectangular synergy at $\ell = n+1$ in (20),

$$\mathcal{S}^{n+k}(U'|\theta') = \textstyle\sum_{u=1}^{n+1} S^{n+k}(r'_u|\theta') = \sum_{u=1}^{n+1} S^{n+k}(i'_u, j'_{n+1-u}, i'_{n+1}, j'_{n+2-u}|\theta') > 0$$

Likewise, NAM uniquely optimal for subsets $\boldsymbol{i}''$ and $\boldsymbol{j}''$ at θ'' implies $\mathcal{S}^{n+k}(U''|\theta'') < 0$.

Part (c): $\mathcal{S}^{n+k}(K^G|\theta') > 0$ for $K^G \equiv U' \cap U''$. First, $U' = \cup_{u=1}^n (i'_u, j'_{n+1-u}, i'_{n+1}, j'_{n+1})$, i.e., a union of rectangles with fixed northeast (Figure 17, panel 3). Likewise, we have $U'' \equiv \cup_{u=1}^n r''_u$. Since $(\boldsymbol{i}', \boldsymbol{j}') \leq (\boldsymbol{i}'', \boldsymbol{j}'')$ (part (a)), if $(i,j) \in U' \setminus U'' = U' \setminus K^G$ (orange shade, Figure 17, panel 2), then $(i'_{u^*}, j'_{n+1-u^*}) \leq (i,j)$, and $i \leq i''_{u^*}$ or $j \leq j''_{n+1-u^*}$, with at least one strict, at some u^*. So couple (i,j) is below the meet of the θ'' matched NAM pair $(i''_{u^*}, j''_{n+2-u^*})$ and $(i''_{u^*+1}, j''_{n+1-u^*})$. As rectangular synergy is upcrossing in types, $s_{ij}(\theta'') < 0$. Then $s_{ij}(\theta') < 0$, as synergy is upcrossing in θ. Then $\mathcal{S}^{n+k}(U' \setminus K^G|\theta') < 0$, as this holds for all $(i,j) \in U' \setminus K^G$. As summed rectangular synergy is additive and $\mathcal{S}^{n+k}(U'|\theta') > 0$ (part (b)), $\mathcal{S}^{n+k}(K^G|\theta') = \mathcal{S}^{n+k}(U'|\theta') - \mathcal{S}^{n+k}(U' \setminus K^G|\theta') > 0$.

Part (d): $\mathcal{S}^{n+k}(K^G|\theta'') < 0$. Since $(\boldsymbol{i}', \boldsymbol{j}') \leq (\boldsymbol{i}'', \boldsymbol{j}'')$ (part (a)), define rectangles $K^Y \equiv (i''_1, j'_{n+1}, i'_{n+1}, j''_{n+1})$ and $K^P \equiv (i'_{n+1}, j''_1, i''_{n+1}, j'_{n+1})$ (resp., yellow and pink regions, Figure 17, panel 2). Then $U'' \setminus K^G = K^Y \cup K^P$. As summed rectangular synergy is linear:

$$\mathcal{S}^{n+k}(K^G|\theta) = \mathcal{S}^{n+k}(U''|\theta) - \mathcal{S}^{n+k}(K^Y|\theta) - \mathcal{S}^{n+k}(K^P|\theta) \tag{21}$$

Rectangle K^Y is above the rectangle defined by the θ' PAM pair (i'_1, j'_n) and (i'_{n+1}, j'_{n+1}). So $\mathcal{S}^{n+k}(K^Y|\theta'') > 0$, as summed rectangular synergy is upcrossing on rectangles and θ. Likewise, K^P is above the rectangle defined by the θ' PAM pair (i'_n, j'_1) and (i'_{n+1}, j'_{n+1}). So $\mathcal{S}^{n+k}(K^P|\theta'') > 0$. Then $\mathcal{S}^{n+k}(K^G|\theta'') < 0$, as $\mathcal{S}^{n+k}(U''|\theta'') < 0$ (part (b)) and (21).

Since $\mathcal{S}^{n+k}(K^G|\theta') > 0$ (part (c)), we cannot have $(\mathcal{S}^{n+k}(K|\theta'), \mathcal{S}^{n+k}(K|\theta'')) \in \hat{\mathcal{D}}_{n+k}$; and thus, by part (a) we have contradicted dyads $(\mathcal{S}^{n+1}(K|\theta'), \mathcal{S}^{n+1}(K|\theta'')) \in \mathcal{D}^*_{n+1}$, and thus conclude that NAM at θ'' and NAM* at θ' is impossible.[23] □

C.2 Proof of Proposition 2 for a Continuum of Types

Step 1. *Uniquely optimal finite type matchings exist for a payoff perturbation with summed rectangular synergy upcrossing in θ.*

Proof: Let $\mathcal{X}^n = \{x_1^n, \ldots, x_n^n\}$ and $\mathcal{Y}^n = \{y_1^n, \ldots, y_n^n\}$ be equal quantile increments, with $G(x_1^n) = H(y_1^n) = 1/n$ and $G(x_i^n) = G(x_{i-1}^n) + 1/n$ and $H(y_j^n) = H(y_{j-1}^n) + 1/n$. Let G^n and H^n be cdfs on $[0,1]$, stepping by $1/n$ at $\mathcal{X}^n$ and $\mathcal{Y}^n$ (resp.). Put $f_{ij}^n(\theta) = \phi(x_i^n, y_j^n|\theta)$. The set $\mathcal{M}^n(\theta)$ of pure optimal matchings is non-empty, by Lemma 2.

Since unique optimal matchings are pure, we restrict to pure matchings. These are uniquely defined by the male partner vector $\mu = (\mu_1, \ldots, \mu_n)$. Call the pure matching $\hat{M}$ *lexicographically higher* than M *iff* its male partner vector $\hat{\mu}$ lexicographically dominates μ. Let $\bar{M}^n(\theta)$ (resp. $\bar{\mu}^n(\theta)$) be the optimal pure matching highest in the lexicographic order, and $\underline{M}^n(\theta)$ (resp. $\underline{\mu}^n(\theta)$) the lowest. Easily, each is well-defined.

[23] This last step assumes upcrossing synergy sums on connected *join semi-lattices* (sets that contain the join of any pair of elements). All of our results only require this weaker time series assumption.

Fix $\theta'' \succ \theta'$. Let $\iota(j) = \bar{\mu}^n_j(\theta') - 1$ and pick $\varepsilon > 0$. Perturb synergy down at θ':

$$s^{n\varepsilon}_{ij}(\theta') \equiv s_{ij}(\theta') - \varepsilon^j \mathbb{1}_{(i,j)=(\iota(j),j)} \tag{22}$$

We prove that $\bar{M}^n(\theta')$ is uniquely optimal at θ' for any production function with ε-perturbed synergy (22), for all small $\varepsilon > 0$. Similar logic will prove that $\underline{M}^n(\theta'')$ is uniquely optimal at θ'' with $s^{n\varepsilon}_{ij}(\theta'') \equiv s_{ij}(\theta'') + \varepsilon^j \mathbb{1}_{(i,j)=(\underline{\mu}^n_j(\theta''),j)}$ for all small $\varepsilon > 0$.

Pick a matching M that is not optimal at $\varepsilon = 0$. Since $\bar{M}^n(\theta')$ is optimal at $\varepsilon = 0$, $\bar{M}^n(\theta')$ yields a higher payoff than M for all small $\varepsilon > 0$.

As $\bar{\mu}^n(\theta')$ is the lexicographically highest optimal matching at θ', another optimal μ obeys $(\bar{\mu}^n_1(\theta'), \ldots, \bar{\mu}^n_{\ell-1}(\theta')) = (\mu_1, \ldots, \mu_{\ell-1})$, and first diverges at $\bar{\mu}^n_\ell(\theta') > \mu_\ell$, for some woman $\ell < n$. Using $M_{ij} = \sum_{k=1}^{j} \mathbb{1}_{\mu_k \le i}$, equation (4), and (22), the payoff $\bar{M}^n(\theta')$ exceeds that of $M \in \mathcal{M}^n(\theta')$ by $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s^{n\varepsilon}_{ij}(\theta')\left[\bar{M}^n_{ij}(\theta') - M_{ij}\right]$. This expands to:

$$\sum_{j=1}^{n-1} \varepsilon^j \left[M_{\iota(j)j} - \bar{M}^n_{\iota(j)j}(\theta')\right] \;=\; \varepsilon^\ell + \sum_{j=\ell+1}^{n-1} \varepsilon^j \sum_{k=\ell+1}^{j} \left[\mathbb{1}_{\mu_k \le \iota(j)} - \mathbb{1}_{\bar{\mu}^n_k \le \iota(j)}\right]$$

Altogether, $\lim_{\varepsilon \to 0} \varepsilon^{-\ell} \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s^{n\varepsilon}_{ij}(\theta')\left[\bar{M}^n_{ij}(\theta') - M_{ij}\right] = 1 > 0$. □

Step 2. *If $\theta'' \succ \theta'$, then $\bar{M}^n(\theta'') \succeq_{PQD} \underline{M}^n(\theta')$ for all n.*

Proof: Since $S^{n\varepsilon}(r|\theta)$ is continuous in ε, there exists $\hat{\varepsilon}_n > 0$ such that, for all $r = (i_1, j_1, i_2, j_2)$ and $0 \le \varepsilon < \hat{\varepsilon}_n$, if $S^{n0}(r|\theta) \lesseqgtr 0$ then $S^{n\varepsilon}(r|\theta) \lesseqgtr 0$. By the contrapositives:

$$S^{n\varepsilon}(r|\theta) \ge 0 \;\Rightarrow\; S^{n0}(r|\theta) \ge 0 \quad \text{and} \quad S^{n\varepsilon}(r|\theta) \le 0 \;\Rightarrow\; S^{n0}(r|\theta) \le 0. \tag{23}$$

We claim that $S^{n\varepsilon}(r|\theta)$ *is strictly upcrossing in r for all $0 < \varepsilon < \hat{\varepsilon}_n$.* For if not, then $S^{n\varepsilon}(r''|\theta) \le 0 \le S^{n\varepsilon}(r'|\theta)$ for some $r'' \succ_{NE} r'$. But then $S^{n0}(r''|\theta) \le 0 \le S^{n0}(r'|\theta)$ by (23), contradicting $S^{n0}(r|\theta)$ strictly upcrossing in r, as follows from Step 1.

Continuum summed rectangular synergy is upcrossing in θ by assumption; and thus, finite summed rectangular synergy $\sum_{k=1} S^{n0}(r_k|\theta)$ for all finite approximations. Then, perturbed summed rectangular synergy $\sum_{k=1} S^{n\varepsilon}(r_k|\theta)$ is upcrossing in θ, since synergy $s^{n\varepsilon}_{ij}(\theta')$ is non-increasing in ε and $s^{n\varepsilon}_{ij}(\theta'')$ is non-decreasing in ε by construction (22).

So for $\varepsilon \in (0, \hat{\varepsilon}_n)$, rectangular synergy $S^{n\varepsilon}(r|\theta)$ is strictly upcrossing in r and summed rectangular synergy $\sum_{k=1} S^{n\varepsilon}(r_k|\theta)$ upcrossing in θ, for couple sets $K \subseteq \mathbb{Z}^2_n$. Given $\bar{M}^n(\theta'), \underline{M}^n(\theta'')$ uniquely optimal, $\underline{M}^n(\theta'') \succeq_{PQD} \bar{M}^n(\theta')\ \forall n$, by Proposition 2. □

Step 3. *There exists a subsequence of matchings $\{M^{n_k}(\theta)\}$ that converges to an optimal matching in the continuum model.*

Proof: Define step function $\phi^n(x, y|\theta) = f^{n\varepsilon_n}_{ij}(\theta)$ for $(x, y) \in [x^n_{i-1}, x^n_i) \times [y^n_{j-1}, y^n_j)$, where $\varepsilon_n = \hat{\varepsilon}_n/n$. Then $\{G^n\}$ and $\{H^n\}$ weakly converge to G and H as $n \to \infty$, while ϕ^n uniformly converges to ϕ. By Theorem 5.20 in Villani (2008), their optimal matching cdfs have a limit point $M^\infty(\theta)$ optimal in the continuum model.[24] □

Step 4. *$M^\infty(\theta'') \succeq_{PQD} M^\infty(\theta')$ for all $\theta'' \succeq \theta'$*

Proof: Fix $\theta'' \succeq \theta'$, and let $\{n_k\}$ be a subsequence along which the sequence of finite type matchings $\{M^{n_k}(\theta')\}$ converges to $M^\infty(\theta')$, as defined in Step 3. Now, since cdfs $\{G^{n_k}\}$ and $\{H^{n_k}\}$ weakly converge to G and H, and $\phi^{n_k}(x, y|\theta'')$ converges uniformly to $\phi(x, y|\theta'')$, there exists a subsequence $\{n_{k_\ell}\}$ of $\{n_k\}$, along which the sequence of finite type matchings $\{M^{n_{k_\ell}}(\theta'')\}$ converges to $M^\infty(\theta'')$ by Theorem 5.20 in Villani (2008). Further, by Step 2, $M^{n_{k_\ell}}(\theta'') \succeq_{PQD} M^{n_{k_\ell}}(\theta')$. But then, the limits must be ordered $M^\infty(\theta'') \succeq_{PQD} M^\infty(\theta')$ by Theorem 9.A.2.a in Shaked and Shanthikumar (2007). □

C.3 Marginal Rectangular Synergy: Proof of Proposition 3

We assume marginal rectangular synergy is upcrossing in types. The steps for downcrossing marginal rectangular synergy are symmetric. We use the relationship:

$$S(x_1, x_2, y_1, y_2|\theta) = \int_{x_1}^{x_2} \Delta_x(x|y_1, y_2, \theta)dx = \int_0^1 \Delta_x(x|y_1, y_2, \theta)\mathbb{1}_{x\in[x_1,x_2]}dx \tag{24}$$

Step 1. *If marginal rectangular synergy is strictly upcrossing, then rectangular synergy is strictly upcrossing.*

Proof: We prove the continuum case, which implies the finite type result. Any indicator function $\mathbb{1}_{x\in[x_1,x_2]}$ is log-supermodular function in (x, x_1) and (x, x_2).[25] By Karlin and Rubin's classic 1956 result, if $\Delta_x(x|y_1, y_2, \theta)$ is upcrossing in x, then the last integral in (24) is upcrossing in x_1 and x_2, and so in (x_1, x_2). Symmetrically, rectangular synergy is upcrossing in (y_1, y_2) when the y-marginal rectangular synergy is upcrossing in y. Altogether, rectangular synergy S is upcrossing in types if both MPIs are upcrossing.

Now assume $\Delta_x(x|y_1, y_2)$ is strictly upcrossing; and so, if $S(x'_1, y_1, x'_2, y_2) = 0$ then $\Delta_x(x'_1|y_1, y_2) < 0 < \Delta_x(x'_2|y_1, y_2)$. So $S_{x_1}(x'_1, y_1, x'_2, y_2) = -\Delta_x(x'_1|y_1, y_2) > 0$ and $S_{x_2}(x'_1, y_1, x'_2, y_2) = \Delta_x(x'_2|y_1, y_2) > 0$. Then $S(x''_1, y_1, x''_2, y_2) > 0$ for all $(x''_1, x''_2) > (x'_1, x'_2)$. By symmetric reasoning, S strictly upcrosses in (y_1, y_2). □

Step 2. *The optimal matching is unique in the continuum type model.*

[24] Namely: Fix a sequence $\{\phi_k\}$ of continuous and uniformly bounded production functions converging uniformly to ϕ. Let $\{G_k\}$ and $\{H_k\}$ be cdf sequences and M_k an optimal matching for ϕ, given G_k and H_k. If G_k and H_k weakly converge to G and H, then some subsequence of $\{M_k\}$ weakly converges to a matching M^* optimal for ϕ, G, and H.

[25] $\phi(x, y) \geq 0$ is *log-supermodular* (LSPM) if $\phi(x', y')\phi(x'', y'') \geq \phi(x', y'')\phi(x'', y')$ for all $x' \leq x''$ and $y' \leq y''$. Easily, we can check that the indicator is LSPM: If $x \in [x_1, x_2]$ and $x' \in [x'_1, x'_2]$ then $\max(x, x') \in [\max(x_1, x'_1), \max(x_2, x'_2)]$ and $\min(x, x') \in [\min(x_1, x'_1), \min(x_2, x'_2)]$.

Proof: By Theorem 5.1 in Ahmad, Kim, and McCann (2011), there is a unique optimal matching when: (*i*) G is absolutely continuous, (*ii*) ϕ is C^2, and (*iii*) the critical points of (what they call a "twist difference") $\phi(x, y_2) - \phi(x, y_1)$ include at most one local max and one local min, for all y_1, y_2. Our continuum types model imposes (*i*) and (*ii*). We claim that (*iii*) follows from marginal rectangular synergy $\Delta_x(x|y_1, y_2) \equiv \phi_1(x, y_2) - \phi_1(x, y_1)$ strictly upcrossing in x, for $y_2 > y_1$. In particular, if $y_2 > y_1$, then $\Delta_x(x|y_1, y_2)$ is upcrossing in x, and any critical point of the twist difference is a global minimum. Similarly, then any critical point is a global maximum if $y_2 < y_1$. □

Step 3. *Sorting increases in θ.*

Proof: Propositions 2 and 3 share the time series assumption. By Step 1, the cross-sectional premise of Proposition 3 implies the cross-sectional premise of Proposition 2. Finally, the optimal matching is generically unique for any finite type model and is unique for continuum type models by Step 2. By Proposition 2, sorting rises in θ. □

C.4 Increasing Sorting: Proof of Proposition 4

Finite Types Proof. We verify the premise of Proposition 2. First, by Theorem 1, total synergy $\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}(\theta)\mathbb{1}_{(i,j)\in Z}$ on any set of couples $Z \subseteq \mathbb{Z}_n^2$ is upcrossing in the parameter $t = \theta$. Thus, summed rectangular synergy $\sum_k S(r_k|\theta)$ is upcrossing in θ for any non-overlapping set of rectangles $\{r_k\}$. Next, rectangular synergy $S(r|\theta) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}(\theta)\mathbb{1}_{(i,j)\in r}$ is upcrossing in r by Theorem 1 with $t = r \in \mathbb{R}^4$. By a similar proof to footnote 25, the indicator function $\mathbb{1}_{(i,j)\in r}$ is LSPM in (i, j, r), since a rectangle r is a sublattice.[26] Then $s_{ij}(\theta)\mathbb{1}_{(i,j)\in r}$ obeys inequality (8) in $z = (i, j)$ and r, since $s_{ij}(\theta)$ obeys (8) for fixed θ. Rectangular synergy upcrosses in r by Theorem 1. □

Continuum of Types Proof. We apply Proposition 3. By Theorem 1, total synergy $\int_Z \phi_{12}(x, y|\theta)dxdy$ is upcrossing in $t = \theta$ for any measurable set $Z \subseteq [0, 1]^2$. Thus, summed rectangular synergy $\sum_k S(R_k|\theta)$ is upcrossing in θ for any non-overlapping set of rectangles $\{R_k\}$. Next, the x-marginal rectangular synergy $\int \phi_{12}(x, y)\mathbb{1}_{y\in[y_1,y_2]}dy$ is strictly upcrossing in x. Let $x'' > x'$. Posit for a contradiction:

$$\int \phi_{12}(x'', y)\mathbb{1}_{y\in[y_1,y_2]}dy \leq 0 \leq \int \phi_{12}(x', y)\mathbb{1}_{y\in[y_1,y_2]}dy \tag{25}$$

As synergy $\phi_{12}(x, y)$ is strictly upcrossing in x and y, by (25), there exist zeros $y', y'' \in (y_1, y_2)$ such that $\phi_{12}(x', y) \lesseqgtr 0$ for $y \lesseqgtr y'$ and $\phi_{12}(x'', y) \lesseqgtr 0$ for $y \lesseqgtr y''$. Easily, these

[26] Theorem 1 assumes $t \in \mathcal{T}$, a poset. Here we exploit the fact that the space of rectangular *sets* of couples is a sublattice of $\mathbb{Z}^2$, even though the PQD order on *distributions* over couples is not a lattice.

zeros are ordered $y'' < y'$. But then inequalities in (25) are simultaneously impossible, for:

$$\begin{aligned} 0 &\leq \int \phi_{12}(x', y)\mathbb{1}_{y\in[y_1,y_2]}dy < \int \phi_{12}(x', y)\mathbb{1}_{y\in[y_1,y'']}\mathbb{1}_{y\in[y',y_2]}dy \\ \Rightarrow\ 0 &< \int \phi_{12}(x'', y)\mathbb{1}_{y\in[y_1,y'']}\mathbb{1}_{y\in[y',y_2]}dy < \int \phi_{12}(x'', y)\mathbb{1}_{y\in[y_1,y_2]}dy \end{aligned}$$

by Theorem 1, since $\int \phi_{12}(x, y)\lambda(y)dy$ is upcrossing in $t = x$ for any non-negative $\lambda(y)$ — because $\phi_{12}(x, y)$ is proportionately upcrossing in types and upcrossing in y. □

C.5 Type Distribution Shifts: Proof of Corollary 2

Throughout, we WLOG assume types shift up in the parameter θ.

Summed Rectangular Quantile Synergy is Upcrossing in θ. In fact, we make the stronger claim that total quantile synergy (10) is upcrossing in θ on any measurable set of quantile pairs $Z \subseteq [0, 1]^2$. In the continuum type model:

$$\Upsilon(\theta) \equiv \int\int \varphi_{12}(p, q|\theta)\mathbb{1}_{(p,q)\in Z}dpdq = \int\int \phi_{12}(x, y)\mathbb{1}_{(G(x|\theta),H(y|\theta))\in Z}dxdy$$

by the change of variables $x = X(p, \theta)$ and $y = Y(q, \theta)$ (equivalently, $p = G(x|\theta)$ and $q = H(y|\theta)$); and thus, $dx = X_p dp$ and $dy = Y_q dq$. Since distributions G and H fall in θ, the cdf associated with pdf $\lambda(x, y|\theta) \equiv \mathbb{1}_{(G(x|\theta),H(y|\theta))\in Z}/[\int\int \mathbb{1}_{(G(x|\theta),H(y|\theta))\in Z}dxdy]$ is stochastically increasing in θ. And thus, since $\phi_{12}(x, y)$ is strictly increasing:

$$0 \leq \Upsilon(\theta) \Rightarrow 0 \leq \int\int \phi_{12}(x, y)\lambda(x, y|\theta)dxdy \leq \int\int \phi_{12}(x, y)\lambda(x, y|\theta')dxdy \Rightarrow 0 \leq \Upsilon(\theta')$$

Identical yields total synergy upcrossing in θ on any set of couples with finite types.

Case (a): Quantile Rectangular Synergy is Upcrossing. Because types $X(p, \theta)$ and $Y(q, \theta)$ are non-decreasing in the quantiles p and q, *quantile rectangular synergy* $S(X(p_1, \theta), Y(q_1, \theta), X(p_2, \theta), Y(q_2, \theta))$ upcrosses in (p_1, q_1, p_2, q_2). Hence, quantile sorting increases in θ by Proposition 2.

Case (b): Quantile Marginal Rectangular Synergy Strictly Upcrosses. Non-decreasing synergy is proportionately upcrossing; and thus $\Delta_x(x|y_1, y_2)$ strictly upcrosses in x as shown in §C.4. Given $G(x|\theta)$ absolutely continuous $X_p > 0$; and so,

$$\Delta_p(p|q_1, q_2, \theta) = \Delta_x(X(p, \theta)|Y(q_1, \theta), Y(q_2, \theta))X_p(p, \theta)$$

is strictly upcrossing in p. Similarly, $\Delta_q(q|p_1, p_2, \theta)$ is strictly upcrossing in q. All told, we've seen that quantile sorting increases in θ, by Step 1 and Proposition 3. □

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D Nowhere Decreasing Optimizers

The space of matching cdf's is not a lattice, since the meet and the join are not defined for arbitrary matchings.[27] The matching problem (3) does not have a lattice constraint or an objective function that is quasi-supermodular in the control: standard monotone comparative static results (e.g. Milgrom and Shannon (1994)) do not apply. The next section presents a general comparative result static for single-crossing functions on partially ordered sets (*posets*) without assuming a well-defined meet or join.[28] We then apply this result to our sorting model to get a nowhere decreasing sorting result.

D.1 Nowhere Decreasing Optimizers for Arbitrary Posets

Let Z and Θ be posets. The correspondence $\zeta : \Theta \to Z$ is *nowhere decreasing* if $z_1 \in \zeta(\theta_1)$ and $z_2 \in \zeta(\theta_2)$ with $z_1 \succeq z_2$ and $\theta_2 \succeq \theta_1$ imply $z_2 \in \zeta(\theta_1)$ and $z_1 \in \zeta(\theta_2)$.

Notably, any partial order $\succeq$ induces a complete (nowhere decreasing) order $\succeq^*$ such that $B \succeq^* A$ if $B = A$ or it is not true that $A \succeq B$. Since the domain of any complete order is a lattice, we can apply standard monotone logic, which we next do.

Theorem 3 (Nowhere Decreasing Optimizers)**.** *Let $F : Z \times \Theta \mapsto \mathbb{R}$, where Z and Θ are posets, and let $Z' \subseteq Z$. If $\max_{z \in Z'} F(z, \theta)$ exists for all θ and F is single crossing in (z, θ), then $\mathcal{Z}(\theta|Z') \equiv \arg\max_{z \in Z'} F(z, \theta)$ is nowhere decreasing in θ for all Z'. If $\mathcal{Z}(\theta|Z')$ is nowhere decreasing in θ for all $Z' \subseteq Z$, then $F(z, \theta)$ is single crossing.*

($\Rightarrow$): If $\theta_2 \succeq \theta_1$, $z_1 \in \mathcal{Z}(\theta_1)$, $z_2 \in \mathcal{Z}(\theta_2)$, and $z_1 \succeq z_2$, optimality and single crossing give:

$$F(z_1, \theta_1) \geq F(z_2, \theta_1) \quad \Rightarrow \quad F(z_1, \theta_2) \geq F(z_2, \theta_2) \quad \Rightarrow \quad z_1 \in \mathcal{Z}(\theta_2)$$

Now assume $z_2 \notin \mathcal{Z}(\theta_1)$. By optimality and single crossing, we get the contradiction:

$$F(z_1, \theta_1) > F(z_2, \theta_1) \quad \Rightarrow \quad F(z_1, \theta_2) > F(z_2, \theta_2) \quad \Rightarrow \quad z_2 \notin \mathcal{Z}(\theta_2)$$

($\Leftarrow$): If F is not single crossing, then for some $z_2 \succeq z_1$ and $\theta_2 \succeq \theta_1$, either: (i) $F(z_2, \theta_1) \geq F(z_1, \theta_1)$ and $F(z_2, \theta_2) < F(z_1, \theta_2)$; or, (ii) $F(z_2, \theta_1) > F(z_1, \theta_1)$ and $F(z_2, \theta_2) \leq F(z_1, \theta_2)$. Let $Z' = \{z_1, z_2\}$. In case (i), $z_2 \in \mathcal{Z}(\theta_1|Z')$ and $z_1 = \mathcal{Z}(\theta_2|Z')$ precludes $\mathcal{Z}(\theta|Z')$

[27] As shown in Proposition 4.12 in Müller and Scarsini (2006): If M dominates PAM2 and PAM4, then $M(2,1) \geq 1/3$ and $M(1,2) \geq 1/3$, but $M(1,1) = 0$ if NAM1 and NAM3 dominate M. So then $M(2,2) = 2/3$, but then NAM1 cannot PQD dominate M.

[28] This may be a known result. We include it for completeness, and as we cannot find any reference.

nowhere decreasing in θ, since $z_2 \notin \mathcal{Z}(\theta_2|Z')$. In case (ii), $z_2 = \mathcal{Z}(\theta_1|Z')$ and $z_1 \in \mathcal{Z}(\theta_2|Z')$ precludes $\mathcal{Z}(\theta|Z')$ nowhere decreasing in θ, since $z_1 \notin \mathcal{Z}(\theta_1|Z')$. □

D.2 Nowhere Decreasing Sorting

Sorting is nowhere decreasing in θ if the matching never falls in the PQD order. So for all $\theta_2 \succeq \theta_1$, if $M_1 \in \mathcal{M}^*(\theta_1)$ and $M_2 \in \mathcal{M}^*(\theta_2)$ are ranked $M_1 \succeq_{PQD} M_2$, then we have $M_2 \in \mathcal{M}^*(\theta_1)$ and $M_1 \in \mathcal{M}^*(\theta_2)$. We say that *weighted synergy is upcrossing*[29] in θ if the following is upcrossing in θ:

- $\int \phi_{12}(x,y|\theta)\lambda(x,y)dxdy$ for all nonnegative (measurable)[30] functions λ on $[0,1]^2$
- $\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} s_{ij}(\theta)\lambda_{ij}$ for all positive weights $\lambda \in \mathbb{R}_+^{(n-1)^2}$

We first present the continuum analogue of the finite match output formula (4).[31]

Lemma 3 (Continuum Types). *Given type intervals $\mathcal{I} \equiv [0,1]$ and $\mathcal{J} \equiv (0,1]$, then:*

$$\int_{\mathcal{I}^2} \phi(x,y)M(dx,dy) = \int_{\mathcal{I}} \phi(x,1)G(dx) - \int_{\mathcal{J}} \phi_2(1,y)H(y)dy + \int_{\mathcal{J}^2} \phi_{12}(x,y)M(x,y)dxdy$$

Proof: If ψ is C^1 on $[0,1]$ and Γ is a cdf on $[0,1]$, integration by parts yields:

$$\int_{[0,1]} \psi(z)\Gamma(dz) = \psi(1)\Gamma(1) - \int_{(0,1]} \psi'(z)\Gamma(z)dz \tag{26}$$

where the interval $(0,1]$ accounts for the possibility that Γ may have a mass point at 0. Since $M(dx,y) \equiv M(y|x)G(dx)$ for a conditional matching cdf $M(y|x)$, we have:

$$M(x,y) \equiv \int_{[0,x]} M(y|x')G(dx') \tag{27}$$

By Theorem 34.5 in Billingsley (1995) and then in sequence (26), (27) and Fubini's

[29] Let Z be a partially ordered set. The function $\sigma : Z \mapsto \mathbb{R}$ is *upcrossing* if $\sigma(z) \geq (>)0$ implies $\sigma(z') \geq (>)0$ for $z' \succeq z$, *downcrossing* if $-\sigma$ is upcrossing. Similarly, σ is strictly upcrossing if $\sigma(z) \geq 0$ implies $\sigma(z') > 0$ for all $z' \succ z$, with strictly downcrossing defined analoguously.

[30] To save space, we henceforth assume measurable sets for integrals whenever needed.

[31] Equation (9) in Cambanis, Simons, and Stout (1976) reduces to our formula when output is C^2. We present our simpler proof for the C^2 case for completeness.

Theorem, (26), the objective function $\int_{[0,1]^2} \phi(x,y)M(dx,dy)$ in (3) equals:

$$\begin{aligned}
& \int_{[0,1]} \int_{[0,1]} \phi(x,y)M(dy|x)G(dx) \\
= & \int_{[0,1]} \phi(x,1)G(dx) - \int_{[0,1]} \int_{(0,1]} \phi_2(x,y)M(y|x)dyG(dx) \\
= & \int_{[0,1]} \phi(x,1)G(dx) - \int_{(0,1]} \left[\phi_2(1,y)M(1,y) - \int_{(0,1]} \phi_{12}(x,y)M(x,y)dx\right] dy
\end{aligned}$$

which easily reduces to the desired expression, using $M(1,y) = H(y)$. □

Theorem 4. *Sorting is nowhere decreasing in θ if weighted synergy is upcrossing in θ, and thus if synergy is nondecreasing in θ. Also, if sorting is nowhere decreasing in θ for all type distributions G, H, then any rectangular synergy is upcrossing in θ.*

Proof of (a): First, $M' \succeq_{PQD} M$ iff $\lambda \equiv M' - M \geq 0$. As weighted synergy upcrosses:

$$\begin{gathered}
\textstyle\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}(\theta)(M'_{ij} - M_{ij}) \geq (>)\ 0 \Rightarrow \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{ij}(\theta')(M'_{ij} - M_{ij}) \geq (>)\ 0 \\
\textstyle\int_{(0,1]^2} \phi_{12}(\cdot|\theta)(M' - M) \geq (>)\ 0 \ \Rightarrow\ \int_{(0,1]^2} \phi_{12}(\cdot|\theta')(M' - M) \geq (>)\ 0
\end{gathered} \tag{28}$$

Thus, match output is single crossing in (M, θ) by (4) (for finite types) and Lemma 3 for continuum types. Then the optimal matching $\mathcal{M}^*(\theta)$ (in the space of feasible matchings $\mathcal{M}(G,H)$) is nowhere decreasing in the state θ, by Theorem 3.

Proof of (b): Assume two women (x_1, x_2) and men (y_1, y_2), and that $S(R|\theta)$ is not upcrossing in θ, i.e. for some $\theta'' \succeq \theta'$ and rectangle $R = (x_1, y_1, x_2, y_2)$, we have $S(R|\theta'') \leq 0 \leq S(R|\theta')$ with one inequality strict. These inequalities imply that NAM optimal at θ'' and PAM optimal at θ', and either NAM is uniquely optimal at θ'' or PAM is uniquely optimal at θ'. Either case precludes nowhere decreasing sorting. □

Easily, weighted synergy is upcrossing in θ if synergy is non-decreasing in θ. Thus:

Corollary 3 (Cambanis, Simons, and Stout (1976))**.** *Sorting is nowhere decreasing in θ if synergy is non-decreasing in θ.*

E Omitted Proofs for Economic Applications in §7

1. Diminishing Returns: Let $R(z|\theta) \equiv -z\psi''(z|\theta)/\psi'(z|\theta)$. Synergy is then:

$$\phi_{12}(x,y|\theta) = \psi'(xy|\theta)\left[\frac{\psi''(xy|\theta)xy}{\psi'(xy|\theta)} + 1\right] \equiv \psi'(xy|\theta)(1 - R(xy|\theta)) \tag{29}$$

By assumption $\psi' > 0$ and $R(xy|\theta)$ is decreasing in x, y, and θ. Thus, synergy strictly upcrosses in x, y, and θ. Further, $\psi'(xy|\theta)$ is smoothly LSPM in (x, y, θ), since

$$[\log(\psi'(xy|\theta))]_x = \frac{y\psi''(xy|\theta)}{\psi'(xy|\theta)} = -x^{-1}R(xy|\theta)$$

is increasing in y and θ by $R(z|\theta)$ decreasing in z and θ. Altogether, synergy (29) is the product of a positive smoothly LSPM function and an increasing function; and thus, synergy is proportionately upcrossing. So sorting increases in θ by Proposition 4.

2. Weakest to Strongest Link: We verify the premise of Proposition 3 to prove that sorting sorting increases in ρ for $\phi(x, y) = \psi(q(x, y))$ as in §7.2. Symmetric steps generalize this result for any $\psi'' < 0 < \psi'$, obeying $2\psi''(q) + q\psi'''(q) \leq 0$.

$$\phi_{12}(x, y) = \frac{q_1(x, y)q_2(x, y)}{q(x, y)}\left[(1 + \rho)(\alpha - 2\beta q(x, y)) - 2\beta q(x, y)\right] \tag{30}$$

Step 1. *Marginal rectangular synergy is strictly downcrossing in types.*

Proof: Since $q(x, y)$ increases in (x, y) and falls in ρ, the bracketed term in (30) falls in (x, y) and rises in ρ. Thus, synergy (30) is upcrossing in ρ and is strictly downcrossing in (x, y). Further, since $q_1(x, y)q_2(x, y)/q(x, y)$ is LSPM in (x, y) when $\rho \geq 0$, synergy is proportionately downcrossing in (x, y). So, marginal rectangular synergy is downcrossing in types, by Theorem 1. Finally, marginal rectangular synergy is strictly downcrossing in (x, y) by the proof logic after inequality (25) in Appendix C.4. □

Step 2. *Summed rectangular synergy is upcrossing in ρ.*

Proof: Since $\phi_{12}(x, y) = \phi_{12}(y, x)$, weighted synergy $\int_{[0,1]^2} \phi_{12}\hat{\lambda}$ is upcrossing in ρ for all weighting functions $\hat{\lambda}$, *iff* $\int_0^1 \int_0^x \phi_{12}(x, y)\lambda(x, y)dxdy$ is upcrossing in ρ for all weighting functions λ. Now use change of variable $y = kx$ to get:

$$\int_0^1 \int_0^x \phi_{12}(x, y)\lambda(x, y)dydx = 2\int_0^1 \int_0^1 x\phi_{12}(x, kx)\lambda(x, kx)dkdx$$

Let $x\phi_{12}(x, kx) = \sigma_A(k, \rho)\sigma_B(x, k, \rho)$, where $\sigma_A \equiv xq_1(x, kx)q_2(x, kx)/q(x, kx)$ and σ_B is the bracketed term in (30) evaluated at $y = kx$. Routine algebra yields $\sigma_A(k, \rho)$ LSPM in (k, ρ), while $\sigma_B(x, k, \rho)$ is decreasing in (x, k) and increasing in ρ. Altogether, $\sigma_A\sigma_B$ is proportionately upcrossing in (x, k, ρ). As synergy is also upcrossing in ρ by Step 1, so is weighted synergy, by Theorem 1 — as is summed rectangular synergy. □

3. Nowhere Decreasing Sorting in Kremer and Maskin (1996):
We prove (11): *sorting is nowhere decreasing in θ and nowhere increasing in $\varrho = -\rho$.*

Step 1. *PAM is not optimal if $\varrho > (1-2\theta)^{-1}$, and is uniquely optimal for $\varrho < (1-2\theta)^{-1}$.*

Proof: In a unisex model, PAM is optimal iff the symmetric rectangular synergy $S(x,x,y,y)$ is globally positive. Its sign is constant along any ray $y = kx$, and proportional to:

$$s(k) \equiv 2^{\frac{1-2\theta}{\varrho}}(1+k) - 2k^{\theta}(1+k^{\varrho})^{\frac{1-2\theta}{\varrho}} \tag{31}$$

Since $s(1) = s'(1) = 0$, $s''(1) \propto (1+\varrho(2\theta-1))$, and $\theta \in [0, 1/2]$, we have $s(k) < 0$ close to $k = 1$ precisely when $\varrho > (1-2\theta)^{-1} \geq 1$. In this case, the symmetric rectangular synergy is negative in a cone around the diagonal, and PAM fails.

Conversely, posit $\varrho < (1-2\theta)^{-1}$. Then $s(k) > 0$ for all $k \in [0,1]$. Since $S(x,x,y,y)$ is symmetric about $y = x$, it is globally positive and PAM is uniquely optimal. □

Step 2. *If $\varrho \geq (1-2\theta)^{-1}$ then weighted synergy is upcrossing in θ, downcrossing in ϱ.*

Proof: Change variables $y = kx$. If $\Delta(k) = \int_0^1 \lambda(x,kx)dx$, weighted synergy is

$$\int\int \phi_{12}(x,y)\lambda(x,y)dydx = 2\int_0^1\int_0^1 x\phi_{12}(x,kx)\lambda(x,kx)dkdx = \int_0^1 \sigma(k,\theta,\varrho)\Delta(k)dk$$

where $\sigma = \sigma_A\sigma_B$ for $\sigma_A \equiv 2k^{\theta-1}(1+k^{\varrho})^{\frac{1-2\theta-2\varrho}{\varrho}}$ and $\sigma_B \equiv \theta(1-\theta)(1+k^{2\varrho}) + (1-\varrho+2\theta(\theta-1+\varrho))k^{\varrho}$. As $\varrho \geq (1-2\theta)^{-1}$, $\sigma_A > 0$ is LSPM in (k,θ,ϱ), σ_B is increasing in $(\theta,-k,-\varrho)$ for $k \in [0,1]$. So $\sigma = \sigma_A\sigma_B$ is proportionately downcrossing in (k,θ) and $(k,-\varrho)$. Weighted synergy is upcrossing in θ, downcrossing in ϱ, by Theorem 1. □

Step 3. *Sorting is nowhere decreasing in θ and nowhere increasing in ϱ.*

Proof: Pick $\theta'' > \theta'$. If $\varrho < (1-2\theta'')^{-1}$, then PAM is uniquely optimal at θ'' (Step 1) and sorting increases from θ' to θ''. If $\varrho \geq (1-2\theta'')^{-1}$, then $\varrho > (1-2\theta')^{-1}$ and weighted synergy is upcrossing on $[\theta',\theta'']$ (Step 2) and sorting is non-decreasing (Proposition 4).

Now pick any θ and $\varrho'' > \varrho'$. If $\varrho' < (1-2\theta)^{-1}$, then PAM is uniquely optimal at ϱ' (Step 1) and sorting is decreasing from ϱ' to ϱ''. If, instead, $\varrho' \geq (1-2\theta)^{-1}$, then, necessarily, $\varrho'' > (1-2\theta)^{-1}$, weighted synergy is downcrossing from ϱ' to ϱ'' (Step 2) and sorting is non-increasing in ϱ, by Proposition 4. □